International Journal of Theoretical Physics, Vol. 40, No. 12, December 2001 (© 2001)

Essentials of Classical Brane Dynamics[1]

Brandon Carter[2]

Received March 10, 2001

This paper provides a self-contained overview of the geometry and dynamics of relativistic brane models, of the category that includes point particle, string, and membrane representations for phenomena that can be considered as being confined to a worldsheet of the corresponding dimension (respectively one, two, and three) in a thin limit approximation in an ordinary 4-dimensional spacetime background. This category also includes "brane world" models that treat the observed universe as a 3-brane in 5 or higher dimensional background. The first sections are concerned with purely kinematic aspects: it is shown how, to second differential order, the geometry (and in particular the inner and outer curvature) of a brane worldsheet of arbitrary dimension is describable in terms of the first, second, and third fundamental tensor. The later sections show how—to lowest order in the thin limit—the evolution of such a brane worldsheet will always be governed by a simple tensorial equation of motion whose left hand side is the contraction of the relevant surface stress tensor $\overline{\boldsymbol{T}}^{\mu\nu}$ with the (geometrically defined) second fundamental tensor $\boldsymbol{K}_{\mu\nu}{}^{\rho}$, while the right hand side will simply vanish in the case of free motion and will otherwise be just the orthogonal projection of any external force density that may happen to act on the brane.

1. INTRODUCTION

This paper is an updated version of the first part of a course originally presented at a school on "Formation and Interactions of Topological Defects" (Carter, 1995). In preparation for the more specific study of strings in the later sections, this first part was intended as an introduction to the systematic study, in a classical relativistic framework, of "branes," meaning physical models in which the relevant fields are confined to supporting worldsheets of lower dimension than the background spacetime. The original version was motivated mainly by applications in which the background spacetime dimension was only 4, but the approach described here is particularly effective for the higher dimensional backgrounds that have very recently become the subject of intensive investigation by cosmological theorists.

[1] Contribution to proc. Peyresq 5 meeting (June, 2000): "Quantum spacetime, brane cosmology, and stochastic effective theories."

[2] D.A.R.C., Observatoire de Paris, 92 Meudon, France.

0020-7748/01/1200-2099/0 © 2001 Plenum Publishing Corporation

Although not entirely new (Dirac, 1962; Howe and Tucker, 1977), the development of classical brane dynamics had lingered at a rather immature stage (compared with the corresponding quantum theory (Achúcarro *et al.*, 1987) which had been stimulated by the rise of "superstring theory"), the main motivation for relatively recent work (Carter, 1990) on classical relativistic brane theory having been its application to vacuum defects produced by the Kibble mechanism (Kibble, 1976), particularly when of composite type as in the case of cosmic strings attached to external domain walls (Vilenkin and Everett, 1982) and of cosmic strings carrying internal currents of the kind whose likely existence was first proposed by Witten (1985) and whose potential cosmological importance, particularly due to the prolific formation of vortons (Davis and Shellard, 1989a), has only recently begun to be generally recognized (Brandenberger *et al.*, 1996). However, interest in the subject has suddenly received a substantial boost from an essentially different quarter.

Following the recent incorporation of 10-dimensional "superstring theory" into 11-dimensional "M theory," the situation has, however, been radically changed in the last couple of years by an upsurge (Binetruy *et al.*, 2000; Chamblin and Gibbons, 2000; Chamblin *et al.*, 2000) of interest in what has come has to be known as "brane world" theory, according to which our observed 4-dimensional universe is to be considered as some kind of brane within a higher dimensional background that is known in this context as the "bulk." Although they are adequate for cases with codimension 1 (which in the "brane world" context means the most commonly considered case (Bowcock *et al.*, 2000; Langlois *et al.*, 2000; Maartens, 2000; Mennim and Battye, 2000; Shiromizu *et al.*, 2000) for which the "bulk" dimension is only 5) traditional methods of analysis have been less satisfactory for cases with codimension 2 or more. The advantage, for such cases, of the more efficient formalism presented here has already been decisively demonstrated within the framework of an ordinary 4-dimensional spacetime background, notably in the context of divergent self interactions of cosmic strings (Carter and Battye, 1998) for which previous methods had provided what turned out to have been misleading results. The superiority of the present approach should be even more overwhelming for the treatment of "brane world" scenarios involving a "bulk" having 6 dimensions (Gherhgetta and Shaposhnikov, 2000) or more.

Before the presentation of the generic dynamic laws governing the evolution of a brane worldsheet (including allowance for the possibility that it may form the boundary of a higher dimensional brane worldsheet) the first sections of this paper provide a recapitulation of the essential differential geometric machinery (Carter, 1992a,b) needed for the analysis of a timelike worldsheet of dimension d say in a background space time manifold of dimension n. At this stage no restriction will be imposed on the curvature of the metric—which will as usual be represented with respect to local background coordinates x^μ ($\mu = 0, \ldots, n-1$) by its components

$\boldsymbol{g}_{\mu\nu}$—though it will be postulated to be flat, or at least stationary or conformally flat, in many of the applications to be discussed later.

2. THE FIRST FUNDAMENTAL TENSOR

The development of geometrical intuition and of computationally efficient methods for use in string and membrane theory has been hampered by a tradition of publishing results in untidy, highly gauge dependent, notation (one of the causes being the undue influence still exercised by Eisenhart's obsolete treatise "Riemannian Geometry" (Eisenhart, 1926). For the intermediate steps in particular calculations it is of course frequently useful and often indispensable to introduce specifically adapted auxiliary structures, such as curvilinear worldsheet coordinates σ^i ($i = 0, \ldots, d-1$) and the associated bitensorial derivatives

$$x^{\mu}{}_{,i} = \frac{\partial x^{\mu}}{\partial \sigma^{i}}, \tag{1}$$

or specially adapted orthonormal frame vectors, consisting of an internal subset of vectors $\iota_A{}^{\mu}$ ($A = 0, \ldots, d-1$) tangential to the worldsheet and an external subset of vectors $\lambda_X{}^{\mu}$ ($X = 1, \ldots, n-d$) orthogonal to the worldsheet, as characterized by

$$\iota_A{}^{\mu}\iota_{B\mu} = \eta_{AB}, \qquad \iota_A{}^{\mu}\lambda_{X\mu} = 0, \qquad \lambda_X{}^{\mu}\lambda_{Y\mu} = \delta_{XY}, \tag{2}$$

where η_{AB} is a fixed d-dimensional Minkowski metric and the Kronecker matrix δ_{XY} is a fixed $(n-d)$-dimensional Cartesian metric. Even in the most recent literature there are still (under Eisenhart's uninspiring influence) many examples of insufficient effort to sort out the messy clutter of indices of different kinds (Greek or Latin, early or late, small or capital) that arise in this way by grouping the various contributions into simple tensorially covariant combinations. Another inconvenient feature of many publications is that results have been left in a form that depends on some particular gauge choice (such as the conformal gauge for internal string coordinates) which obscures the relationship with other results concerning the same system but in a different gauge.

The strategy adopted here (Stachel, 1980) aims at minimizing such problems (they can never be entirely eliminated) by working as far as possible with a single kind of tensor index, which must of course be the one that is most fundamental, namely that of the background coordinates, x^{μ}. Thus, to avoid dependence on the internal frame index A (which is lowered and raised by contraction with the fixed d-dimensional Minkowski metric η_{AB} and its inverse η^{AB}) and on the external frame index X (which is lowered and raised by contraction with the fixed $(n-d)$-dimensional Cartesian metric δ_{XY} and its inverse δ^{XY}), the separate internal frame vectors $\iota_A{}^{\mu}$ and external frame vectors $\lambda_X{}^{\mu}$ will as far as possible be eliminated in favor of frame gauge independent combinations such as the unit tangent d-vector (i.e., antisymmetric contravariant d-index tensor) with spacetime components given, for a p-brane with $p = d - 1$, by

$$\mathcal{E}^{\mu\dots\sigma} = (p+1)!\, \iota_0{}^{[\mu} \dots \iota_p{}^{\sigma]}, \tag{3}$$

which is useful for many purposes but has the inconvenient feature of being not strictly tensorial but only pseudo-tensorial (since its sign is dependent on an orientation convention that would be reversed if the ordering of the frame vectors were subject to an odd permutation) as well as having the property (which is particularly awkward for higher dimensional applications) that the number of component indices involved is dimension dependent. These inconvenient features can, however, be avoided in many contexts by following an approach based on what we refer to as the (first) fundamental tensor of the worldsheet, which is definable as the (rank d) operator of tangential projection onto the worldsheet. This fundamental tensor, which we shall denote here by $\eta^\mu{}_\nu$, is given, along with the complementary (rank $n-d$) operator $\perp^\mu{}_\nu$ of projection orthogonal to the world sheet, by

$$\eta^\mu{}_\nu = \iota_A{}^\mu\, \iota^A{}_\nu, \qquad \perp^\mu{}_\nu = \lambda_X{}^\mu\, \lambda^X{}_\nu. \tag{4}$$

The same principle (of minimization of the use of auxiliary gauge dependent reference systems) applies to the avoidance of unnecessary involvement of the internal coordinate indices which are lowered and raised by contraction with the induced metric on the worldsheet as given by

$$\gamma_{ij} = \boldsymbol{g}_{\mu\nu} x^\mu{}_{,i} x^\nu{}_{,j}, \tag{5}$$

and with its contravariant inverse γ^{ij}. After being cast (by index raising if necessary) into its contravariant form, any internal coordinate tensor can be directly projected onto a corresponding background tensor in the manner exemplified by the intrinsic metric itself, which gives

$$\eta^{\mu\nu} = \gamma^{ij} x^\mu{}_{,i} x^\nu{}_{,j}, \tag{6}$$

thus providing an alternative (more direct) prescription for the fundamental tensor that was previously introduced via the use of the internal frame in (4). This approach also provides a direct prescription for the orthogonal projector that was introduced via the use of an external frame in (4) but that is also obtainable immediately from (6) as

$$\perp^\mu{}_\nu = \boldsymbol{g}^\mu{}_\nu - \eta^\mu{}_\nu. \tag{7}$$

As well as having the separate operator properties

$$\eta^\mu{}_\rho\, \eta^\rho{}_\nu = \eta^\mu{}_\nu, \qquad \perp^\mu{}_\rho \perp^\rho{}_\nu = \perp^\mu{}_\nu \tag{8}$$

the tensors defined by (6) and (7) will evidently be related by the conditions

$$\eta^\mu{}_\rho \perp^\rho{}_\nu = 0 = \perp^\mu{}_\rho\, \eta^\rho{}_\nu. \tag{9}$$

3. THE INNER AND OUTER CURVATURE TENSORS

In so far as we are concerned with tensor fields such as the frame vectors whose support is confined to the d-dimensional worldsheet, the effect of Riemannian covariant differentation ∇_μ along an arbitrary directions on the background spacetime will not be well defined, only the corresponding tangentially projected differentiation operation

$$\overline{\nabla}_\mu \stackrel{\text{def}}{=} \eta^\nu{}_\mu \nabla_\nu, \tag{10}$$

being meaningful for them, as for instance in the case of a scalar field φ for which the tangentially projected gradient is given in terms of internal coordinate differentiation simply by $\overline{\nabla}^\mu \varphi = \gamma^{ij} x^\mu{}_{,i}\, \varphi_{,ji}$.

An irreducible basis for the various possible covariant derivatives of the frame vectors consists of the internal rotation pseudo-tensor $\rho_\mu{}^\nu{}_\rho$ and the external rotation (or "twist") pseudo-tensor $\varpi_\mu{}^\nu{}_\rho$ as given by

$$\rho_\mu{}^\nu{}_\rho = \eta^\nu{}_\sigma\, \iota^A{}_\rho \overline{\nabla}_\mu \iota_A{}^\sigma = -\rho_{\mu\rho}{}^\nu, \quad \varpi_\mu{}^\nu{}_\rho = \perp^\nu{}_\sigma \lambda^X{}_\rho \overline{\nabla}_\mu \lambda_X{}^\sigma = -\varpi_{\mu\rho}{}^\nu, \tag{11}$$

together with their mixed analogue $\boldsymbol{K}_{\mu\nu}{}^\rho$ which is obtainable in a pair of equivalent alternative forms given by

$$\boldsymbol{K}_{\mu\nu}{}^\rho = \perp^\rho{}_\sigma\, \iota^A{}_\nu \overline{\nabla}_\mu \iota_A{}^\sigma = -\eta^\sigma{}_\nu\, \lambda_X{}^\rho \overline{\nabla}_\mu \lambda^X{}_\sigma. \tag{12}$$

The reason for qualifying the fields (11) as "pseudo-tensors" is that although they are tensorial in the ordinary sense with respect to changes of the background coordinates x^μ they are not geometrically well defined just by the geometry of the worldsheet but are gauge dependent in the sense of being functions of the choice of the internal and external frames $\iota_A{}^\mu$ and $\lambda_X{}^\mu$. The gauge dependence of $\rho_\mu{}^\nu{}_\rho$ and $\varpi_\mu{}^\nu{}_\rho$ means that both of them can be set to zero at any chosen point on the worldsheet by choice of the relevant frames in its vicinity. However, the condition for it to be possible to set these pseudo-tensors to zero throughout an open neigborhood is the vanishing of the curvatures of the corresponding frame bundles as characterized with respect to the respective invariance subgroups SO$(1, d-1)$ and SO$(n-d)$ into which the full Lorentz invariance group SO$(1, n-1)$ is broken by the specification of the d-dimensional worldsheet orientation. The inner curvature that needs to vanish for it to be possible for $\rho_\mu{}^\nu{}_\rho$ to be set to zero in an open neighborhood is of Riemannian type, and is obtainable (by a calculation of the type originally developed by Cartan that was made familiar to physicists by Yang Mills theory) as (Carter, 1992a)

$$\boldsymbol{R}_{\kappa\lambda}{}^\mu{}_\nu = 2\eta^\mu{}_\sigma \eta^\tau{}_\mu \eta^\pi{}_{[\lambda} \overline{\nabla}_{\kappa]} \rho_\pi{}^\sigma{}_\tau + 2\rho_{[\kappa}{}^{\mu\pi} \rho_{\lambda]\pi\nu}, \tag{13}$$

while the outer curvature that needs to vanish for it to be possible for the "twist" tensor $\varpi_\mu{}^\nu{}_\rho$ to be set to zero in an open neighborhood is of a less familiar type that is given (Carter, 1992a) by

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2\perp^{\mu}{}_{\sigma}\perp^{\tau}{}_{\mu}\,\eta^{\pi}{}_{[\lambda}\overline{\nabla}_{\kappa]}\varpi_{\pi}{}^{\sigma}{}_{\tau} + 2\varpi_{[\kappa}{}^{\mu\pi}\varpi_{\lambda]\pi\nu}. \tag{14}$$

The frame gauge invariance of the expressions (13) and (14)—which means that $\boldsymbol{R}_{\kappa\lambda}{}^{\mu}{}_{\nu}$ and $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$ are unambiguously well defined as tensors in the strictest sense of the word—is not immediately obvious from the foregoing formulae, but it is made manifest in the the alternative expressions given in section 6.

4. THE SECOND FUNDAMENTAL TENSOR

Another, even more fundamentally important, gauge invariance property that is not immediately obvious from the traditional approach—as recapitulated in the preceeding section is—that of the entity $\boldsymbol{K}_{\mu\nu}{}^{\rho}$ defined by the mixed analogue (12) of (11), which (unlike $\rho_{\mu}{}^{\nu}{}_{\rho}$ and $\varpi_{\mu}{}^{\nu}{}_{\rho}$, but like $\boldsymbol{R}_{\kappa\lambda}{}^{\mu}{}_{\nu}$ and $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$) is in fact a geometrically well defined tensor in the strict sense. To see that the formula (12) does indeed give a result that is frame gauge independent, it suffices to verify that it agrees with the alternative—manifestly gauge independent definition (Carter, 1990)

$$\boldsymbol{K}_{\mu\nu}{}^{\rho} \overset{\text{def}}{=} \eta^{\sigma}{}_{\nu}\overline{\nabla}_{\mu}\eta^{\rho}{}_{\sigma}. \tag{15}$$

whereby the entity that we refer to as the second fundamental tensor is constructed directly from the the first fundamental tensor $\eta^{\mu\nu}$ as given by (6).

Since this second fundamental tensor, $\boldsymbol{K}_{\mu\nu}{}^{\rho}$ will play a very important role throughout the work that follows, it is worthwhile to linger over its essential properties. To start with it is to be noticed that a formula of the form (15) could of course be meaningfully applied not only to the fundamental projection tensor of a d-surface, but also to any (smooth) field of rank-d projection operators $\eta^{\mu}{}_{\nu}$ as specified by a field of arbitrarily orientated d-surface elements. What distinguishes the integrable case, i.e., that in which the elements mesh together to form a well-defined d-surface through the point under consideration, is the condition that the tensor defined by (15) should also satisfy the Weingarten identity

$$\boldsymbol{K}_{[\mu\nu]}{}^{\rho} = 0 \tag{16}$$

(where the square brackets denote antisymmetrization), this symmetry property of the second fundamental tensor being derivable (Carter, 1990, 1992a) as a version of the well known Frobenius theorem. In addition to this nontrivial symmetry property, the second fundamental tensor is also obviously tangential on the first two indices and almost as obviously orthogonal on the last, i.e.,

$$\perp^{\sigma}{}_{\mu}\,\boldsymbol{K}_{\sigma\nu}{}^{\rho} = \boldsymbol{K}_{\mu\nu}{}^{\sigma}\,\eta_{\sigma}{}^{\rho} = 0. \tag{17}$$

The second fundamental tensor $\boldsymbol{K}_{\mu\nu}{}^{\rho}$ has the property of fully determining the tangential derivatives of the first fundamental tensor $\eta^{\mu}{}_{\nu}$ by the formula

$$\overline{\nabla}_{\mu}\eta_{\nu\rho} = 2\,\boldsymbol{K}_{\mu(\nu\rho)} \tag{18}$$

(using round brackets to denote symmetrization) and it can be seen to be characterizable by the condition that the orthogonal projection of the acceleration of any tangential unit vector field $\boldsymbol{u}^{\mu}$ will be given by

$$\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}\ \boldsymbol{K}_{\mu\nu}{}^{\rho} = \perp^{\rho}{}_{\mu}\ \dot{\boldsymbol{u}}^{\mu}, \qquad \dot{\boldsymbol{u}}^{\mu} = \boldsymbol{u}^{\nu}\nabla_{\nu}\,\boldsymbol{u}^{\mu}. \tag{19}$$

In cases for which we need to use the d-index surface element pseudo-tensor $\mathcal{E}^{\mu\ldots\sigma}$ given for the d-dimensional worldsheet of the p-brane by (3), it will be useful to have the relevant surface derivative formula which takes the form

$$\overline{\nabla}_{\lambda}\mathcal{E}^{\mu\ldots\sigma} = (-1)^{p}(p+1)\mathcal{E}^{\nu[\mu\ldots}\boldsymbol{K}_{\lambda\nu}{}^{\sigma]}, \tag{20}$$

in which it is to be recalled that $p = d-1$. (This expression corrects what is, as far as I am aware, the only wrongly printed formula in the more complete analysis (Carter, 1992a) on which this presentation is based: the factor $(-1)^{p}$ was inadvertently omitted in the relevant formula (B9), which is thus valid as printed only for a worldsheet of odd dimension $d = p+1$.)

5. EXTRINSIC CURVATURE VECTOR AND CONFORMATION TENSOR

It is very practical for a great many purposes to introduce the extrinsic curvature vector K^{μ}, defined as the trace of the second fundamental tensor, which is automatically orthogonal to the worldsheet,

$$\boldsymbol{K}^{\mu} \stackrel{\text{def}}{=} \boldsymbol{K}^{\nu}{}_{\nu}{}^{\mu}, \qquad \eta^{\mu}{}_{\nu}\,\boldsymbol{K}^{\nu} = 0. \tag{21}$$

It is useful for many specific purposes to work this out in terms of the intrinsic metric γ_{ij} and its determinant $|\gamma|$. It suffices to use the simple expression $\overline{\nabla}^{\mu}\varphi = \gamma^{ij}x^{\mu}{}_{,i}\varphi_{,j}$ for the tangentially projected gradient of a scalar field φ on the worldsheet, but for a tensorial field (unless one is using Minkowski coordinates in a flat spacetime) there will also be contributions involving the background Riemann–Christoffel connection

$$\Gamma_{\mu}{}^{\nu}{}_{\rho} = \boldsymbol{g}^{\nu\sigma}\big(\boldsymbol{g}_{\sigma(\mu,\rho)} - \tfrac{1}{2}\,\boldsymbol{g}_{\mu\rho,\sigma}\big). \tag{22}$$

The curvature vector is thus obtained in explicit detail as

$$\boldsymbol{K}^{\nu} = \overline{\nabla}_{\mu}\eta^{\mu\nu} = \frac{1}{\sqrt{\|\gamma\|}}\big(\sqrt{\|\gamma\|}\gamma^{ij}x^{\nu}{}_{,i}\big)_{,j} + \gamma^{ij}x^{\mu}{}_{,i}x^{\rho}{}_{,j}\Gamma_{\mu}{}^{\nu}{}_{\rho}. \tag{23}$$

This last expression is technically useful for certain specific computational purposes, but it must be remarked that much of the literature on cosmic string dynamics has been made unnecessarily heavy to read by a tradition of working all the time with long strings of nontensorial terms such as those on the right of (23) rather than taking advantage of such more succinct tensorial expressions as the preceeding formula $\overline{\nabla}_\mu \eta^{\mu\nu}$. As an alternative to the universally applicable tensorial approach advocated here, there is of course another more commonly used method of achieving succinctness in particular circumstances, which is to sacrifice gauge covariance by using specialized kinds of coordinate system. In particular for the case of a string, i.e. for a 2-dimensional worldsheet, it is standard practise to use conformal coordinates σ^0 and σ^1 so that the corresponding tangent vectors $\dot{x}^\mu = x^\mu{}_{,0}$ and $x'^\mu = x^\mu{}_{,1}$ satisfy the restrictions $\dot{x}^\mu x'_\mu = 0$, $\dot{x}^\mu \dot{x}_\mu + x'^\mu x'_\mu = 0$, which implies $\sqrt{\|\gamma\|} = x'^\mu x'_\mu = -\dot{x}^\mu \dot{x}_\mu$ so that (23) simply gives $\sqrt{\|\gamma\|}\, \boldsymbol{K}^\nu = x''^\nu - \ddot{x}^\nu + (x'^\mu x'^\rho - \dot{x}^\mu \dot{x}^\rho)\Gamma_\mu{}^\nu{}_\rho$.

The physical specification of the extrinsic curvature vector (21) for a timelike d-surface in a dynamic theory provides what can be taken as the equations of extrinsic motion of the d-surface (Carter, 1990, 1992b), the simplest possibility being the "harmonic" condition $\boldsymbol{K}^\mu = 0$ that is obtained (as will be shown in the following sections) from a surface measure variational principle such as that of the Dirac membrane model (Dirac, 1962), or of the Goto–Nambu string model (Kibble, 1976) whose dynamic equations in a flat background are therefore expressible with respect to a standard conformal gauge in the familiar form $x''^\mu - \ddot{x}^\mu = 0$.

There is a certain analogy between the Einstein vacuum equations, which impose the vanishing of the trace $\mathcal{R}_{\mu\nu}$ of the background spacetime curvature $\mathcal{R}_{\lambda\mu}{}^\rho{}_\nu$, and the Dirac–Gotu–Nambu equations, which impose the vanishing of the trace $\boldsymbol{K}^\nu$ of the second fundamental tensor $\boldsymbol{K}_{\lambda\mu}{}^\nu$. Just as it is useful to separate out the Weyl tensor (Schouten, 1954), i.e., the trace free part of the Ricci background curvature which is the only part that remains when the Einstein vacuum equations are satisfied, so also analogously, it is useful to separate out the trace free part of the second fundamental tensor, namely the extrinsic conformation tensor (Carter, 1992a), which is the only part that remains when equations of motion of the Dirac–Goto–Nambu type are satisfied. Explicitly, the trace free extrinsic conformation tensor $\boldsymbol{C}_{\mu\nu}{}^\rho$ of a d-dimensional imbedding is defined (Carter, 1992a) in terms of the corresponding first and second fundamental tensors $\eta_{\mu\nu}$ and $\boldsymbol{K}_{\mu\nu}{}^\rho$ as

$$\boldsymbol{C}_{\mu\nu}{}^\rho \stackrel{\text{def}}{=} \boldsymbol{K}_{\mu\nu}{}^\rho - \frac{1}{d}\eta_{\mu\nu}\boldsymbol{K}^\rho, \quad \boldsymbol{C}^\nu{}_\nu{}^\mu = 0. \tag{24}$$

Like the Weyl tensor $\mathcal{W}_{\lambda\mu}{}^\rho{}_\nu$ of the background metric (whose definition is given implicitly by (29) below) this conformation tensor has the noteworthy property of being invariant with respect to conformal modifications of the background metric:

$$\boldsymbol{g}_{\mu\nu} \mapsto e^{2\alpha}\, \boldsymbol{g}_{\mu\nu}, \Rightarrow \boldsymbol{K}_{\mu\nu}{}^\rho \mapsto \boldsymbol{K}_{\mu\nu}{}^\rho + \eta_{\mu\nu}\perp^{\rho\sigma}\nabla_\sigma\alpha, \quad \boldsymbol{C}_{\mu\nu}{}^\rho \mapsto \boldsymbol{C}_{\mu\nu}{}^\rho. \tag{25}$$

This formula is useful (Carter *et al.*, 1994) for calculations of the kind undertaken by Vilenkin (1991) in a standard Robertson–Walker type cosmological background, which can be obtained from a flat auxiliary spacetime metric by a conformal transformation for which e^{α} is a time dependent Hubble expansion factor.

6. THE CODAZZI, GAUSS, AND SCHOUTEN IDENTITIES

As the higher order analogue of (15) we can go on to introduce the third fundamental tensor (Carter, 1990) as

$$\Xi_{\lambda\mu\nu}{}^{\rho} \overset{\text{def}}{=} \eta^{\sigma}{}_{\mu}\eta^{\tau}{}_{\nu}\perp^{\rho}{}_{\alpha}\overline{\nabla}_{\lambda}\,\boldsymbol{K}_{\sigma\tau}{}^{\alpha}, \tag{26}$$

which by construction is obviously symmetric between the second and third indices and tangential on all the first three indices. In a spacetime background that is flat (or of constant curvature as is the case for the DeSitter universe model) this third fundamental tensor is fully symmetric over all the first three indices by what is interpretable as the generalized Codazzi identity which is expressible (Carter, 1992a) in a background with arbitrary Riemann curvature $\mathcal{R}_{\lambda\mu}{}^{\rho}{}_{\sigma}$ as

$$\Xi_{\lambda\mu\nu}{}^{\rho} = \Xi_{(\lambda\mu\nu)}{}^{\rho} + \tfrac{2}{3}\eta^{\sigma}{}_{\lambda}\eta^{\tau}{}_{(\mu}\eta^{\alpha}{}_{\nu)}\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha}\perp^{\rho}{}_{\beta} \tag{27}$$

It is to be noted that a script symbol $\mathcal{R}$ is used here in order to distinguish the (n-dimensional) background Riemann curvature tensor from the intrinsic curvature tensor (13) of the (d-dimensional) worldsheet to which the ordinary symbol $\boldsymbol{R}$ has already allocated.

For many of the applications that will follow it will be sufficient just to treat the background spacetime as flat, i.e., to take $\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha} = 0$. At this stage however, we shall allow for an unrestricted background curvature. For $n > 2$, this will be decomposable in terms of its trace free Weyl part $\mathcal{W}_{\mu\nu}{}^{\rho}{}_{\sigma}$ (which as remarked above is conformally invariant) and the corresponding background Ricci tensor and its scalar trace,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu}{}^{\rho}{}_{\nu}, \qquad \mathcal{R} = \mathcal{R}^{\nu}{}_{\nu}, \tag{28}$$

in the form (Schouten, 1954)

$$\mathcal{R}_{\mu\nu}{}^{\rho\sigma} = \mathcal{W}_{\mu\nu}{}^{\rho\sigma} + \tfrac{4}{n-2}\boldsymbol{g}^{[\rho}{}_{[\mu}\mathcal{R}^{\sigma]}{}_{\nu]} - \tfrac{2}{(n-1)(n-2)}\mathcal{R}\,\boldsymbol{g}^{[\rho}{}_{[\mu}\boldsymbol{g}^{\sigma]}{}_{\nu]}, \tag{29}$$

(in which the Weyl contribution can be nonzero only for $n \geq 4$). In terms of the tangential projection of this background curvature, one can evaluate the corresponding internal curvature tensor (13) in the form

$$\boldsymbol{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\,\boldsymbol{K}^{\rho}{}_{[\mu}{}^{\tau}\,\boldsymbol{K}_{\nu]\sigma\tau} + \eta^{\kappa}{}_{\mu}\eta^{\lambda}{}_{\nu}\mathcal{R}_{\kappa\lambda}{}^{\alpha}{}_{\tau}\eta^{\rho}{}_{\alpha}\eta^{\tau}{}_{\sigma}, \tag{30}$$

which is the translation into the present scheme of what is well known in other schemes as the generalized Gauss identity. The much less well known analogue for the (identically trace free and conformally invariant) outer curvature (14) (for which the most historically appropriate name might be argued to be that of Schouten (1954) is given (Carter, 1992a) in terms of the corresponding projection of the background Weyl tensor by the expression

$$\Omega_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\,\boldsymbol{C}_{[\mu}{}^{\tau\rho}\boldsymbol{C}_{\nu]\tau\sigma} + \eta^{\kappa}{}_{\mu}\eta^{\lambda}{}_{\nu}\mathcal{W}_{\kappa\lambda}{}^{\alpha}{}_{\tau}\perp^{\rho}{}_{\alpha}\perp^{\tau}{}_{\alpha}. \tag{31}$$

It follows from this last identity that in a background that is flat or conformally flat (for which it is necessary, and for $n \geq 4$ sufficient, that the Weyl tensor should vanish) the vanishing of the extrinsic conformation tensor $\boldsymbol{C}_{\mu\nu}{}^{\rho}$ will be sufficient (independently of the behavior of the extrinsic curvature vector $\boldsymbol{K}^{\mu}$) for vanishing of the outer curvature tensor $\Omega_{\mu\nu}{}^{\rho}{}_{\sigma}$, which is the condition for it to be possible to construct fields of vectors λ^{μ} orthogonal to the surface and such as to satisfy the generalized Fermi–Walker propagation condition to the effect that $\perp^{\rho}{}_{\mu}\,\overline{\nabla}_{\nu}\lambda_{\rho}$ should vanish. It can also be shown (Carter, 1992a) (taking special trouble for the case $d = 3$) that in a conformally flat background (of arbitrary dimension n) the vanishing of the conformation tensor $\boldsymbol{C}_{\mu\nu}{}^{\rho}$ is always sufficient (though by no means necessary) for conformal flatness of the induced geometry in the imbedding.

7. THE INTERNAL RICCI AND CONFORMAL CURVATURES

The conclusion of the preceding paragraph is an illustration of the critically significant role of the conformation tensor $\boldsymbol{C}_{\mu\nu}{}^{\rho}$ of an imbedding when the background is conformally flat, which suggests that it will be of interest to make a closer examination of its role with respect to the inner curvature, $\boldsymbol{R}_{\kappa\lambda}{}^{\mu}{}_{\nu}$ and more particularly of its tensorially irreducible parts, in this conformally flat case, for which the condition that the background Weyl tensor should vanish is necessary—and for $n \geq 4$ also sufficient (Schouten, 1954)—while when the background dimension is $n = 3$, this condition, namely $\mathcal{W}_{\kappa\lambda}{}^{\mu}{}_{\nu} = 0$, will hold in any case as an identity. This restriction is of course compatible with all the most common kinds of application, in which the background is taken to be not just conformally flat, but flat in the strong sense, which is justifiable at least as a very good approximation in a very wide range of circumstances in which the characteristic length scales of the imbedding will be small compared with those of the background curvature if any. Although it is unnecessary for such cases, we shall nevertheless retain allowance for the possibility of a nonzero background Ricci tensor $\mathcal{R}_{\mu\nu}$ in the formulae that follows since the extra complication involved thereby is only very moderate (compared with what would result if allowance for a nonzero background Weyl tensor were also included).

Leaving aside the trivial (always locally conformally flat) case of a 2-dimensional background, the generalized Gauss relation (30) reduces to the form

$$\boldsymbol{R}_{\kappa\lambda}{}^{\mu}{}_{\nu} = \frac{2}{n-2}\big(\eta_{[\kappa}{}^{\mu}\eta_{\lambda]}{}^{\rho}\eta_{\nu}{}^{\rho} - \eta_{\nu[\kappa}\eta_{\lambda]}{}^{\rho}\eta^{\mu\sigma}\big)\left(\mathcal{R}_{\rho\sigma} - \frac{\mathcal{R}}{2(n-1)}\boldsymbol{g}_{\rho\sigma}\right)$$
$$+ 2\boldsymbol{K}_{[\kappa}{}^{\mu\sigma}\boldsymbol{K}_{\lambda]\nu\sigma} + \eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\nu}\eta^{\mu}{}_{\tau}\eta^{\nu}{}_{\nu}, \tag{32}$$

in which the last term evidently drops out whenever the background Weyl tensor vanishes. Proceeding from this formula by contraction, the internal Ricci tensor is obtained in terms of the irreducible parts $\boldsymbol{K}_{\rho}$ and $\boldsymbol{C}_{\lambda\mu}{}^{\nu}$ of the second fundamental tensor $\boldsymbol{K}_{\mu\nu}{}^{\rho}$ in the form

$$\boldsymbol{R}_{\mu\nu} = \frac{p-2}{n-2}\eta_{\mu}{}^{\rho}\eta_{\nu}{}^{\sigma}\,\mathcal{R}_{\rho\sigma} + \frac{1}{n-2}\left(\eta^{\rho\sigma}\mathcal{R}_{\rho\sigma} - \frac{p-1}{n-1}\mathcal{R}\right)\eta_{\mu\nu}$$
$$+ \frac{p-1}{p^2}\boldsymbol{K}^{\sigma}\boldsymbol{K}_{\sigma}\eta_{\mu\nu} + \frac{p-2}{p}\boldsymbol{C}_{\mu\nu}{}^{\sigma}\,\boldsymbol{K}_{\sigma} - \boldsymbol{C}_{\mu}{}^{\rho\sigma}\boldsymbol{C}_{\nu\rho\sigma} + \mathcal{W}_{\mu\nu}, \tag{33}$$

where the final background Weyl contribution, if any, is given by the expressions

$$\mathcal{W}_{\mu\nu} = \eta_{\mu}{}^{\sigma}\,\eta_{\nu}{}^{\kappa}\,\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\kappa}\eta^{\rho}{}_{\tau} = -\eta_{\mu}{}^{\sigma}\,\eta_{\nu}{}^{\kappa}\,\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\kappa}\perp^{\rho}{}_{\tau}, \tag{34}$$

of which the last version is obtained as a consequence of the tracelessness of the Weyl tensor.

The corresponding Ricci scalar for the internal geometry (whose surface integral in the special case $p = 2$ gives the ordinary Gauss Bonnet type invariant that was mentioned at the end of section 8) is thus finally obtained in the form

$$\boldsymbol{R} = \frac{p-1}{n-2}\left(2\eta^{\rho\sigma}\mathcal{R}_{\rho\sigma} - \frac{p}{n-1}\mathcal{R}\right) + \frac{p-1}{p}\boldsymbol{K}^{\sigma}\boldsymbol{K}_{\sigma} - \boldsymbol{C}_{\lambda\mu}{}^{\nu}\,\boldsymbol{C}^{\lambda\mu}{}_{\nu} + \mathcal{W}, \tag{35}$$

(which corrects a transcription error whereby a factor of two was omitted in the original version (Carter, 1992a) where the final Weyl contribution is just the trace

$$\mathcal{W} = \mathcal{W}^{\nu}{}_{\nu} = \eta^{\sigma\nu}\mathcal{W}_{\rho\sigma}{}^{\rho}{}_{\nu}\eta^{\nu}{}_{\tau} = \perp^{\sigma\nu}\mathcal{W}_{\rho\sigma}{}^{\rho}{}_{\nu}\perp^{\nu}{}_{\tau}, \tag{36}$$

which can be seen to vanish identically unless both the dimension and the codimension of the worldsheet are greater than one, i.e., unless both $p \geq 2$ and $n - p \geq 2$.

For cases in which the imbedded surface has dimension $p \leq 3$, as must always be the case in an ordinary 4-dimensional spacetime background, the specification of the Ricci contribution provides all that is needed to specify the complete inner curvature tensor. However, to fully specify $\boldsymbol{R}_{\kappa\lambda}{}^{\mu}{}_{\nu}$ in higher dimensional cases for which the imbedded surface has dimension $p \geq 4$, it will also be necessary to take account of the generically nonzero conformal curvature term $\mathcal{C}_{\kappa\lambda}{}^{\mu}{}_{\nu}$ that will contribute to the total as given by the internal analogue of (29), namely

$$\boldsymbol{R}_{\mu\nu}{}^{\rho\sigma} = \mathcal{C}_{\mu\nu}{}^{\rho\sigma} + \tfrac{4}{p-2}\eta^{[\rho}{}_{[\mu}\,\boldsymbol{R}^{\sigma]}{}_{\nu]} - \tfrac{2}{(p-1)(p-2)}\boldsymbol{R}\eta^{[\rho}{}_{[\mu}\,\eta^{\sigma]}{}_{\nu]}. \tag{37}$$

The rather greater algebraic effort required to work out this inner conformal curvature contribution is rewarded by the qualitatively tidy form of the result, which (in contrast with the miscellaneous form of the terms assembled in (7) and (35) is homogeneously quadratic in the conformation tensor alone, the contributions of the trace vector $\boldsymbol{K}^{\mu}$ and of the background Ricci tensor $\mathcal{R}_{\mu\nu}$ again (as in (31)) being found to miraculously cancel out altogether, leaving

$$\mathcal{C}_{\kappa\lambda}{}^{\mu\nu} = 2\,\boldsymbol{C}_{[\kappa}{}^{\mu\sigma}\,\boldsymbol{C}_{\lambda]}{}^{\nu}{}_{\sigma} - \tfrac{4}{p-2}\left(\boldsymbol{C}^{\rho[\mu}{}_{\sigma}\eta^{\nu]}{}_{[\kappa}\,\boldsymbol{C}_{\lambda]\rho}{}^{\sigma} + \eta_{[\kappa}{}^{[\mu}\,\mathcal{W}_{\lambda]}{}^{\nu]}\right)$$
$$- \tfrac{2}{(p-2)(p-1)}\eta_{[\kappa}{}^{\mu}\,\eta_{\lambda]}{}^{\nu}\left(\boldsymbol{C}_{\rho\sigma}{}^{\tau}\,\boldsymbol{C}^{\rho\sigma}{}_{\tau} - \mathcal{W}\right)$$
$$+ \eta_{\kappa}{}^{\rho}\,\eta_{\lambda}{}^{\sigma}\,\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\nu}\eta^{\mu}{}_{\tau}\eta^{\nu\nu}. \tag{38}$$

We can thus draw the memorable conclusion that in a conformally flat background the vanishing of the conformation tensor $\boldsymbol{C}^{\mu\nu}{}_{\rho}$ is a sufficient condition not only for (local) outer flatness but also for (local) internal conformal flatness, at least for an imbedded surface with dimension $p \geq 4$. With a little more work (Carter, 1992a) it can be shown that this conclusion also holds for $p = 3$, while it is trivial for the case of a string worldsheet $p = 2$, which is always (locally) conformally flat.

8. THE SPECIAL CASE OF A STRING WORLDSHEET IN 4-DIMENSIONS

The application with which we shall mainly be concerned in the following work will be the case $d = 2$ of a string. An orthonormal tangent frame will consist in this case just of a timelike unit vector, $\iota_0{}^{\mu}$, and a spacelike unit vector, $\iota_1{}^{\mu}$, whose exterior product vector is the frame independent antisymmetric unit surface element tensor

$$\mathcal{E}^{\mu\nu} = 2\iota_0{}^{[\mu}\iota_1{}^{\nu]} = 2(-|\gamma|)^{-1/2}\,x^{[\mu}{}_{,0}\,x^{\nu]}{}_{,1}, \tag{39}$$

whose tangential gradient satisfies

$$\overline{\nabla}_{\lambda}\mathcal{E}^{\mu\nu} = -2\,\boldsymbol{K}_{\lambda\rho}{}^{[\mu}\mathcal{E}^{\nu]\rho}. \tag{40}$$

In this case the inner rotation pseudo-tensor (11) is determined just by a corresponding rotation covector ρ_{μ} according to the specification

$$\rho_{\lambda}{}^{\mu}{}_{\nu} = \tfrac{1}{2}\mathcal{E}^{\mu}{}_{\nu}\rho_{\lambda}, \qquad \rho_{\lambda} = \rho_{\lambda}{}^{\mu}{}_{\nu}\mathcal{E}^{\nu}{}_{\mu}. \tag{41}$$

This can be used to see from (13) that the Ricci scalar,

$$\boldsymbol{R} = \boldsymbol{R}^{\nu}{}_{\nu} \qquad \boldsymbol{R}_{\mu\nu} = \boldsymbol{R}_{\rho\mu}{}^{\rho}{}_{\nu}, \tag{42}$$

of the 2-dimensional worldsheet will have the well known property of being a pure surface divergence, albeit of a frame gauge dependent quantity:

$$\boldsymbol{R} = \bar{\nabla}_\mu(\mathcal{E}^{\mu\nu}\,\rho_\nu). \tag{43}$$

In the specially important case of a string in ordinary 4-dimensional spacetime, i.e. when we have not only $d = 2$ but also $n = 4$, the antisymmetric background measure tensor $\varepsilon^{\lambda\mu\nu\rho}$ can be used to determine a scalar (or more strictly, since its sign is orientation dependent, a pseudo-scalar) magnitude Ω for the outer curvature tensor (14) (despite the fact that its traces are identically zero) according to the specification

$$\Omega = \tfrac{1}{2}\Omega_{\lambda\mu\nu\rho}\,\varepsilon^{\lambda\mu\nu\rho}. \tag{44}$$

Under these circumstances one can also define a "twist" covector ϖ_μ, that is the outer analogue of ρ_μ, according to the specification

$$\varpi_\nu = \tfrac{1}{2}\varpi_\nu{}^{\mu\lambda}\,\varepsilon_{\lambda\mu\rho\sigma}\,\mathcal{E}^{\rho\sigma}. \tag{45}$$

This can be used to deduce from (14) that the outer curvature (pseudo) scalar Ω of a string worldsheet in 4-dimensions has a divergence property of the same kind as that of its more widely known Ricci analogue (43), the corresponding formula being given by

$$\Omega = \bar{\nabla}_\mu(\mathcal{E}^{\mu\nu}\varpi_\nu). \tag{46}$$

It is to be remarked that for a compact spacelike 2-surface the integral of (40) gives the well known Gauss–Bonnet invariant, but that the timelike string worldsheets under consideration here will not be characterized by any such global invariant since they will not be compact (being open in the time direction even for a loop that is closed in the special sense). The outer analogue of the Gauss–Bonnet invariant that arises from (44) for a spacelike 2-surface has been discussed by Penrose and Rindler (1984) but again there is no corresponding global invariant in the necessarily noncompact timelike case of a string worldsheet.

9. REGULAR AND DISTRIBUTIONAL FORMULATIONS OF BRANE ACTION

The term p-brane has come into use (Achúcarro *et al.*, 1987; Bars and Pope, 1988) to describe a dynamic system localized on a timelike support surface of dimension $d = p + 1$, imbedded in a spacetime background of dimension $n > p$. Thus at the low dimensional extreme one has the example of a zero-brane, meaning what is commonly referred to as a "point particle," and of a 1-brane, meaning what is commonly referred to as a "string." At the high dimensional extreme one has the "improper" case of an $(n-1)$-brane, meaning what is commonly referred to as a "medium" (as exemplified by a simple fluid), and of an $(n-2)$-brane, meaning what is commonly referred to as a "membrane" (from which the generic term "brane" is derived). A membrane (as exemplified by a cosmological domain wall) has the special feature of being supported by a hypersurface, and so being able to form a boundary between separate background space time regions; this means that a 2-brane has the status of being a membrane in ordinary 4-dimensional spacetime (with $n = 4$) but not in a higher dimensional (e.g., Kaluza–Klein type) background.

The purpose of the present section is to consider the dynamics not just of an individual brane but of a brane complex or "rigging model" (Carter, 1990) such as is illustrated by the nautical archetype in which the wind—a 3-brane—acts on a boat's sail—a 2-brane—that is held in place by cords—1-branes—which meet at knots, shackles, and pulley blocks that are macroscopically describable as point particles—i.e., 0-branes. In order for a set of branes of diverse dimensions to qualify as a "geometrically regular" brane complex or "rigging system" it is required not only that the support surface of each $(d-1)$-brane should be a smoothly imbedded d-dimensional timelike hyper-surface but also that its boundary, if any, should consist of a disjoint union of support surfaces of an attatched subset of lower dimensional branes of the complex. (For example in order qualify as part of a regular brane complex the edge of a boat's sail cannot be allowed to flap freely but must be attatched to a hem cord belonging to the complex.) For the brane complex to qualify as regular in the strong dynamic sense that will be postulated in the present work, it is also required that a member p-brane can exert a direct force only on an attached $(p-1)$-brane on its boundary or on an attached $(p+1)$-brane on whose boundary it is itself located, though it may be passively subject to forces exerted by a higher dimensional background field. For instance the Peccei–Quin axion model gives rise to field configurations representable as regular complexes of domain walls attached to strings (Shellard, 1990; Sikivie, 1982; Vilenkin, 1982), and a bounded (topological or other) Higgs vortex defect terminated by a pair of pole defects (Copeland *et al.*, 1988; Manton, 1983; Martin, 1996; Nambu, 1977; Vachaspati and Achúcarro, 1991; Vachaspati and Barriola, 1992) may be represented as a regular brane complex consisting of a finite cosmic string with a pair of point particles at its ends, in an approximation neglecting Higgs field radiation. (However, allowance for radiation would require the use of an extended complex including the Higgs medium whose interaction with the string—and a fortiori with the terminating particles—would violate the regularity condition: the ensuing singularities in the back reaction would need to be treated by a renormalization procedure of a kind (Battye and Shellard, 1995, 1996; Dabholkar and Quashnock, 1990; Shellard, 1990) whose development so far has been beset with difficulties in preserving exact local Lorentz invariance, an awkward problem that is beyond the scope of the present paper.

The present section will be restricted to the case of a brane complex that is not only regular in the sense of the preceeding paragraph but that is also pure (or "fine") in the sense that the lengthscales characterizing the internal structure of the (defect or other) localized phenomenon represented by the brane models are short compared with those characterizing the macroscopic variations under consideration so that polarization effects play no role. For instance in the case of a point particle, the restriction that it should be describable as a "pure" zero brane simply means that it can be represented as a simple monopole without any dipole or higher multipole effects. In the case of a cosmic string the use of a "pure" 1-brane description requires that the underlying vortex defect be sufficiently thin compared not only with its total length but also compared with the lengthscales characterizing its curvature and the gradients of any currents it may be carrying. The effect of the simplest kind of curvature corrections beyond this "pure brane" limit has been considered by several authors for strings (Gregory, 1988, 1993; Maeda and Turok, 1988; Polyakov, 1986), domain walls (Barrabès *et al.*, 1994; Carter and Gregory, 1995; Gregory *et al.*, 1991; Silveira and Maia, 1993), and more generally (Arodz *et al.*, 1991; Boisseau and Letelier, 1992; Capovilla and Guven, 1995a; Carter, 1994a; Hartley and Tucker, 1990; Letelier, 1990), but in the rest of this paper, as in the present section, it will be assumed that the ratio of microscopic to macroscopic lengthscales is sufficiently small for description in terms of "pure" p-branes to be adequate.

The present section will not be concerned with the specific details of particular cases but with the generally valid laws that can be derived as Noether identities from the postulate that the model is governed by dynamical laws derivable from a variational principle specified in terms of an action function $\mathcal{I}$. It is however to be emphasized that the validity at a macroscopic level of the laws given here is not restricted to cases represented by macroscopic models of the strictly conservative type directly governed by a macroscopic variational principle. The laws obtained here will also be applicable to classical models of dissipative type (e.g., allowing for resistivity to relative flow by internal currents) as necessary conditions for the existence of an underlying variational description of the microscopic (quantum) degrees of freedom that are allowed for merely as entropy in the macroscopically averaged classical description.

In the case of a brane complex, the total action $\mathcal{I}$ will be given as a sum of distinct d-surface integrals respectively contributed by the various $(d-1)$-branes of the complex, of which each is supposed to have its own corresponding Lagrangian surface density scalar ${}^{(d)}\overline{\mathcal{L}}$ say. Each supporting d-surface will be specified by a mapping $\sigma \mapsto x\{\sigma\}$ giving the local background coordinates x^μ ($\mu = 0, \ldots, n-1$) as functions of local internal coordinates σ^i ($i = 0, \ldots, d-1$). The corresponding d-dimensional surface metric tensor ${}^{(d)}\gamma_{ij}$ that is induced (in the manner described in section 2) as the pull back of the n-dimensional background spacetime metric $\boldsymbol{g}_{\mu\nu}$, will determine the natural surface measure, ${}^{(d)}d\overline{\mathcal{S}}$, in terms of which the total action will be expressible in the form

$$\mathcal{I} = \sum_d \int {}^{(d)}\!d\overline{\mathcal{S}}\, {}^{(d)}\!\overline{\mathcal{L}}, \qquad {}^{(d)}\!d\overline{\mathcal{S}} = \sqrt{\|{}^{(d)}\gamma\|}\, d^d\sigma. \tag{47}$$

As a formal artifice whose use is an unnecessary complication in ordinary dynamical calculations but that can be useful for purposes such as the calculation of radiation, the confined (d-surface supported) but locally regular Lagrangian scalar fields ${}^{(d)}\overline{\mathcal{L}}$ can be replaced by corresponding unconfined, so no longer regular but distributional fields ${}^{(d)}\hat{\mathcal{L}}$, in order to allow the the basic multidimensional action (47) to be represented as a single integral,

$$\mathcal{I} = \int d\mathcal{S} \sum_d {}^{(d)}\hat{\mathcal{L}}, \qquad d\mathcal{S} = \sqrt{\|\boldsymbol{g}\|}\, d^n x. \tag{48}$$

over the n-dimensional background spacetime. In order to do this, it is evident that for each $(d-1)$-brane of the complex the required distributional action contribution ${}^{(d)}\hat{\mathcal{L}}$ must be constructed in terms of the corresponding regular d-surface density scalar ${}^{(d)}\overline{\mathcal{L}}$ according to the presecription that is expressible in standard Dirac notation as

$${}^{(d)}\hat{\mathcal{L}} = \|\boldsymbol{g}\|^{-1/2} \int {}^{(d)}\!d\overline{\mathcal{S}}\;\; {}^{(d)}\overline{\mathcal{L}}\, \delta^n[x - x\{\sigma\}]. \tag{49}$$

10. CURRENT, GENERALIZED VORTICITY, AND STRESS-ENERGY TENSOR

In the kind of model under consideration, each supporting d-surface is supposed to be endowed with its own independent internal field variables which are allowed to couple with each other and with their derivatives in the corresponding d-surface Lagrangian contribution ${}^{(d)}\overline{\mathcal{L}}$, and which are also allowed to couple into the Lagrangian contribution ${}^{(d-1)}\overline{\mathcal{L}}$ on any of its attached boundary $(d-1)$ surfaces, though—in order not to violate the strong dynamic regularity condition—they are not allowed to couple into contributions of dimension $(d-2)$ or lower. As well as involving its own d-brane surface fields and those of any $(d+1)$ brane to whose boundary it may belong, each contribution ${}^{(d)}\overline{\mathcal{L}}$ may also depend passively on the fields of a fixed higher dimensional background. Such fields will of course always include the background spacetime metric $\boldsymbol{g}_{\mu\nu}$ itself. Apart from that, the most commonly relevant kind of background field (the only one allowed for in the earlier analysis, (Carter, 1990)) is a Maxwellian gauge potential $\boldsymbol{A}_\mu$ whose exterior derivative is the automatically "closed" electromagnetic field,

$$\boldsymbol{F}_{\mu\nu} = 2\nabla_{[\mu}\, \boldsymbol{A}_{\nu]}, \qquad \nabla_{[\mu}\, \boldsymbol{F}_{\nu\rho]} = 0. \tag{50}$$

Although many other possibilities can in principle be envisaged, the most commonly relevant generalization, to which for the sake of simplicity the following analysis will be limited, consists of allowance just for another background field of the generic Ramond type (of which the ordinary gauge covector $\boldsymbol{A}_\nu$ is a special single index case) that is important in wide range of applications including the kind of cosmic or superfluid defects for which this work is particularly intended, namely a gauge r-form, i.e., an antisymmetric covariant r-index tensor field with components $\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} = \boldsymbol{A}^{\{r\}}{}_{[\nu\mu\ldots]}$, whose exterior derivative is an automatically closed physical current $(r+1)$-form,

$$\boldsymbol{F}^{\{r+1\}}{}_{\mu\nu\rho\ldots} = (r+1)\,\nabla_{[\mu}\,\boldsymbol{A}^{\{r\}}{}_{\nu\rho\ldots]}, \qquad \nabla_{[\mu}\,\boldsymbol{F}^{\{r+1\}}{}_{\nu\rho\sigma\ldots]} = 0. \tag{51}$$

Just as a Maxwellian gauge tranformation of the form $\boldsymbol{A}_\mu \mapsto \boldsymbol{A}_\mu + \nabla_\mu \alpha$ for an arbitrary scalar α leaves the electromagnetic field (50) invariant, so analogously a Kalb–Ramond gauge transformation $\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} \mapsto \boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} + r!\nabla_{[\mu}\chi_{\nu\ldots]}$ for an arbitrary $(r-1)$-form $\chi_{\mu\ldots}$ leaves the corresponding current $(r+1)$-form (51) invariant.

An example of the kind that is most common in an ordinary 4-dimensional spacetime background is that of a Kalb–Ramond field, meaning a 2-index Ramond field with components $\boldsymbol{A}^{\{2\}}{}_{\mu\nu} = -\boldsymbol{A}^{\{2\}}{}_{\nu\mu}$ for which the corresponding current 3-form $\boldsymbol{F}^{\{3\}}{}_{\mu\nu\rho} = \nabla_\mu\,\boldsymbol{A}^{\{2\}}{}_{\nu\rho} + \nabla_\nu\,\boldsymbol{A}^{\{2\}}{}_{\rho\mu} + \nabla_\rho\,\boldsymbol{A}^{\{2\}}{}_{\mu\nu}$ will just be the dual $\boldsymbol{F}^{\{3\}}{}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\,n^\sigma$ of an ordinary current vector n^μ satisfying a conservation law of the usual type, $\nabla_\mu n^\mu = 0$. Such a Kalb–Ramond representation can be used to provide an elegant variational formulation for ordinary perfect fluid theory (Carter, 1994b) and is particularly convenient for setting up "global" string models of vortices both in a simple cosmic axion or Higgs field (Davis and Shellard, 1989b; Sakellariadou, 1991; Vilenkin, 1987) and in a superfluid (Ben-Ya'acov, 1992) such as liquid Helium-4.

In accordance with the preceeding considerations, the analysis that follows will be based on the postulate that the action is covariantly and gauge invariantly determined by specifying each scalar Lagrangian contribution ${}_{(d)}\overline{\mathcal{L}}$ as a function just of the background fields, $\boldsymbol{A}_\mu$, $\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots}$ and of course $\boldsymbol{g}_{\mu\nu}$, and of any relevant internal fields (which in the simplest nontrivial case—exemplied by string models (Carter, 1989a; Larsen, 1993) of the category needed for the macroscopic description of Witten type (Witten, 1985) superconducting vortices—consist just of a phase scalar φ). In accordance with the restriction that the branes be "pure" or "fine" in the sense explained above, it is postulated that polarization effects are excluded by ruling out couplings involving gradients of the background fields. This means that the effect of making arbitrary infinitesimal "Lagrangian" variations $\underset{\mathrm{L}}{\delta}\,\boldsymbol{A}_\mu$, $\underset{\mathrm{L}}{\delta}\,\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots}$, $\underset{\mathrm{L}}{\delta}\,\boldsymbol{g}_{\mu\nu}$ of the background fields will be to incduce a corresponding variation $\delta\mathcal{I}$ of the action that simply has the form

$$\delta\mathcal{I} = \sum_d \int {}^{(d)}d\overline{\mathcal{S}}\left\{ {}^{(d)}\overline{\boldsymbol{j}}^\mu\,\underset{\mathrm{L}}{\delta}\,\boldsymbol{A}_\mu + \frac{1}{r!}\,{}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots}\,\underset{\mathrm{L}}{\delta}\,\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} + \frac{1}{2}\,{}^{(d)}\overline{\boldsymbol{T}}^{\mu\nu}\,\underset{\mathrm{L}}{\delta}\,\boldsymbol{g}_{\mu\nu}\right\}, \tag{52}$$

provided either that the relevant independent internal field components are fixed or else that the internal dynamic equations of motion are satisfied in accordance with the variational principle stipulating that variations of the relevant independent field variables should make no difference. For each d-brane of the complex, as well as the surface stress momentum energy density tensor ${}^{(d)}\overline{\boldsymbol{T}}^{\mu\nu} = {}^{(d)}\overline{\boldsymbol{T}}^{\nu\mu}$, this partial differentiation formula also implicity specifies the corresponding electromagnetic surface current density vector ${}^{(d)}\overline{\boldsymbol{j}}^{\mu}$, and the (antisymmetric) surface flux r-vector ${}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots} = {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{[\mu\nu\ldots]}$, which is interpretable as vorticity in the 2-index Kalb–Ramond case. These quantities are formally expressible more explicitly as

$$ {}^{(d)}\overline{\boldsymbol{j}}^{\mu} = \frac{\partial\,{}^{(d)}\overline{\mathcal{L}}}{\partial\,\boldsymbol{A}_{\mu}}, \qquad {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots} = r!\,\frac{\partial\,{}^{(d)}\overline{\mathcal{L}}}{\partial\,\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots}}, \tag{53} $$

and

$$ {}^{(d)}\overline{\boldsymbol{T}}^{\mu\nu} = 2\frac{\partial\,{}^{(d)}\overline{\mathcal{L}}}{\partial\,\boldsymbol{g}_{\mu\nu}} + {}^{(d)}\overline{\mathcal{L}}\,{}^{(d)}\eta^{\mu\nu}, \tag{54} $$

of which the latter is obtained using the formula

$$ \underset{\mathrm{L}}{\delta}\big({}^{(d)}d\overline{\mathcal{S}}\big) = \tfrac{1}{2}\,{}^{(d)}\eta^{\mu\nu}\,\big(\underset{\mathrm{L}}{\delta}\ \boldsymbol{g}_{\mu\nu}\big)\,{}^{(d)}d\overline{\mathcal{S}}, \tag{55} $$

where ${}^{(d)}\eta^{\mu\nu}$ is the rank-d fundamental tensor of the d-dimensional imbedding, as defined in the manner described in the preceeding section.

11. CONSERVATION OF CURRENT AND GENERALIZED VORTICITY

The condition that the action be gauge invariant means that if one simply sets $\underset{\mathrm{L}}{\delta}\,\boldsymbol{A}_{\mu} = \nabla_{\mu}\alpha$, $\underset{\mathrm{L}}{\delta}\,\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} = r!\nabla_{[\mu}\chi_{\nu\ldots]}$, $d_{\mathrm{L}}\,\boldsymbol{g}_{\mu\nu} = 0$, for arbitrarily chosen α and $\chi_{\mu\ldots}$ then $\delta\mathcal{I}$ should simply vanish, i.e.,

$$ \sum_{d}\int d\,{}^{(d)}\overline{\mathcal{S}}\left\{{}^{(d)}\overline{\boldsymbol{j}}^{\mu}\nabla_{\mu}\alpha + {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\cdots}\nabla_{\mu}\chi_{\nu\ldots}\right\} = 0. \tag{56} $$

In order for this to be able to hold for an arbitrary scalar field α and an arbitrary $(r-1)$ form χ_{μ} it is evident that the surface current ${}^{(d)}\overline{\boldsymbol{j}}^{\mu}$ and the (generalized vorticity) flux r-vector ${}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\cdots}$ must (as one would anyway expect from the consideration that they depend just on the relevant internal d-surface fields) be purely d-surface tangential, i.e., their contractions with the relevant rank $(n-d)$ orthogonal projector ${}^{(d)}\perp^{\mu}{}_{\nu} = \boldsymbol{g}^{\mu}{}_{\nu} - {}^{(d)}\eta^{\mu}{}_{\nu}$ must vanish:

$$ {}^{(d)}\perp^{\mu}{}_{\nu}\,{}^{(d)}\overline{\boldsymbol{j}}^{\nu} = 0, \qquad {}^{(d)}\perp^{\mu}{}_{\nu}\,{}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\nu\rho\cdots} = 0. \tag{57} $$

Hence, decomposing the full gradient operator ∇_{μ} as the sum of its tangentially projected part ${}^{(d)}\overline{\nabla}_{\mu} = {}^{(d)}\eta^{\nu}{}_{\mu}\nabla_{\nu}$ and of its orthogonally projected part ${}^{(d)}\perp^{\nu}{}_{\mu}\nabla_{\mu}$, and noting that by (57) the latter will give no contribution, one sees that (56) will take the form

$$\sum_d \int {}^{(d)}\! d\overline{\mathcal{S}} \left\{ {}^{(d)}\overline{\nabla}_\mu \left({}^{(d)}\overline{\boldsymbol{j}}^\mu \alpha + {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} \chi_{\nu\dots} \right) \right.$$
$$\left. - \alpha\, {}^{(d)}\overline{\nabla}_\mu \overline{\boldsymbol{j}}^\mu - \chi_{\nu\dots}\, {}^{(d)}\overline{\nabla}_\mu\, {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} \right\} = 0, \tag{58}$$

in which first term of each integrand is a pure surface divergence. Such a divergence can be dealt with using Green's theorem, according to which, for any d-dimensional support surface ${}^{(d)}\overline{\mathcal{S}}$ of a $(d-1)$-brane, one has the identity

$$\int {}^{(d)}\! d\overline{\mathcal{S}}\, {}^{(d)}\overline{\nabla}_\mu\, {}^{(d)}\overline{\boldsymbol{j}}^\mu = \oint {}^{(d-1)}\! d\overline{\mathcal{S}}\, {}^{(d)}\lambda_\mu\, {}^{(d)}\overline{\boldsymbol{j}}^\mu, \tag{59}$$

where is integral on the right is taken over the boundary $(d-1)$-surface $\partial\, {}^{(d)}\overline{\mathcal{S}}$ of ${}^{(d)}\overline{\mathcal{S}}$, and ${}^{(d)}\lambda_\mu$ is the (uniquely defined) unit tangent vector on the d-surface that is directed normally outwards at its $(d-1)$-dimensional boundary. Bearing in mind that a membrane support hypersurface can belong to the boundary of two distinct media, and that for $d \leq n-3$ a d-brane may belong to a common boundary joining three or more distinct $(d+1)$-branes of the complex under consideration, one sees that (58) is equivalent to the condition

$$\sum_p \int {}^{(p)}\! d\overline{\mathcal{S}} \left\{ \alpha \left({}^{(p)}\overline{\nabla}_\mu\, {}^{(p)}\overline{\boldsymbol{j}}^\mu - \sum_{d=p+1} {}^{(d)}\lambda_\mu\, {}^{(d)}\overline{\boldsymbol{j}}^\mu \right) \right.$$
$$\left. + \chi_{\nu\dots} \left({}^{(p)}\overline{\nabla}_\mu\, {}^{(p)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} - \sum_{d=p+1} {}^{(d)}\lambda_\mu\, {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} \right) \right\} = 0, \tag{60}$$

where, for a particular p-dimensionally supported $(p-1)$-brane, the summation "over $d = p+1$" is to be understood as consisting of a contribution from each $(p+1)$-dimensionally supported p-brane attached to it, where for each such p-brane, ${}^{(d)}\lambda_\mu$ denotes the (uniquely defined) unit tangent vector on its $(p+1)$-dimensional support surface that is directed normally towards the p-dimensional support surface of the boundary $(p-1)$-brane. The Maxwell gauge invariance requirement to the effect that (60) should hold for arbitrary α can be seen to entail an electromagnetic charge conservation law of the form

$${}^{(p)}\overline{\nabla}_\mu\, {}^{(p)}\overline{\boldsymbol{j}}^\mu = \sum_{d=p+1} {}^{(d)}\lambda_\mu\, {}^{(d)}\overline{\boldsymbol{j}}^\mu. \tag{61}$$

This can be seen from (59) to be interpretable as meaning that the total charge flowing out of particular $(d-1)$-brane from its boundary is balanced by the total charge flowing into it from any d-branes to which it may be attached. The analogous Ramond gauge invariance requirement that (60) should also hold for an arbitrary $(r-1)$-form $\chi_{\mu\dots}$ can be seen to entail a corresponding (vorticity) flux conservation law of the form

$$ {}^{(p)}\overline{\nabla}_\mu\, {}^{(p)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} = \sum_{d=p+1} {}^{(d)}\lambda_\mu\, {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots}. \tag{62} $$

A more sophisticated but less practical way of deriving the foregoing conservation laws would be to work not from the expression (47) in terms of ordinary surface integrals but instead to use the superficially simpler expression (48) in terms of destributions, which leads to the replacement of (61) by the ultimately equivalent (more formally obvious but less directly meaningful) expression

$$ \nabla_\mu \left(\sum_d {}^{(d)}\hat{\boldsymbol{j}}^\mu \right) = 0 \tag{63} $$

involving the no longer regular but Dirac distributional current ${}^{(d)}\hat{\boldsymbol{j}}^\mu$ that is given in terms of the corresponding regular surface current ${}^{(d)}\overline{\boldsymbol{j}}^\mu$ by

$$ {}^{(d)}\hat{\boldsymbol{j}}^\mu = \|\boldsymbol{g}\|^{-1/2} \int {}^{(d)}d\overline{\mathcal{S}}\, {}^{(d)}\overline{\boldsymbol{j}}^\mu\, \delta^n[x - x\{\sigma\}]. \tag{64} $$

Similarly one can if one wishes rewrite the flux conservation law (62) in the distributional form

$$ \nabla_\mu \left(\sum {}^{(d)}\hat{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} \right) = 0, \tag{65} $$

where the distributional (generalized vorticity) flux ${}^{(d)}\hat{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots}$ is given in terms of the corresponding regular surface flux ${}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots}$ by

$$ {}^{(d)}\hat{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots} = \|\boldsymbol{g}\|^{-1/2} \int {}^{(d)}d\overline{\mathcal{S}}\, {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\dots}\, \delta^n[x - x\{\sigma\}]. \tag{66} $$

It is left as an entirely optional exercise for any readers who may be adept in distribution theory to show how the ordinary functional relationships (61) and (62) can be recovered by integrating out the Dirac distributions in (63) and (65).

12. FORCE AND THE STRESS BALANCE EQUATION

The condition that the hypothetical variations introduced in (52) should be “Lagrangian” simply means that they are to be understood to be measured with respect to a reference system that is comoving with the various branes under consideration, so that their localization with respect to it remains fixed. This condition is necessary for the variation to be meaningly definable at all for a field whose support is confined to a particular brane locus, but in the case of an unrestricted background field one can envisage the alternative possibility of an “Eulerian” variation, meaning one defined with respect to a reference system that is fixed in advance, independently of the localization of the brane complex, the standard example being that of a Minkowski reference system in the case of a background that is flat. In such a case the relation between the more generally meaningful Lagrangian (comoving) variation, denoted by $\underset{\mathrm{L}}{\delta}$, and the corresponding Eulerian (fixed point) variation denoted by $\underset{\mathrm{E}}{\delta}$ say will be given by Lie differentiation with respect to the vector field ξ^μ say that specifies the infinitesimal of the comoving reference system with respect to the fixed background, i.e., one has

$$\underset{\mathrm{L}}{\delta} - \underset{\mathrm{E}}{\delta} = \vec{\xi}\pounds, \tag{67}$$

where the Lie differentiation operator $\vec{\xi}\pounds$ is given for the background fields under consideration here by

$$\vec{\xi}\pounds\, \boldsymbol{A}_\mu = \xi^\sigma \nabla_\sigma\, \boldsymbol{A}_\mu + \boldsymbol{A}_\sigma\, \nabla_\mu \xi^\sigma, \tag{68}$$

$$\vec{\xi}\pounds\, \boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} = \xi^\sigma \nabla_\sigma\, \boldsymbol{A}^{\{r\}}{}_{\mu\nu} + r!\, \boldsymbol{A}^{\{r\}}{}_{\sigma[\nu\ldots}\nabla_{\mu]}\xi^\sigma, \tag{69}$$

$$\vec{\xi}\pounds\, \boldsymbol{g}_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}. \tag{70}$$

This brings us to the main point of this section which is the derivation of the dynamic equations governing the extrinsic motion of the branes of the complex, which are obtained from the variational principle to the effect that the action $\mathcal{I}$ is left invariant not only by infinitesimal variations of the relevant independent intrinsic fields on the support surfaces but also by infinitesimal displacement of the support surfaces themselves. Since the background fields $\boldsymbol{A}_\mu$, $\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots}$, and $\boldsymbol{g}_{\mu\nu}$ are to be considered as fixed, the relevant Eulerian variations simply vanish, and so the resulting Lagrangian variations will be directly identifiable with the corresponding Lie derivatives—as given by (70)—with respect to the generating vector field ξ^μ of the infinitesimal displacement under consideration. The variational principle governing the equations of extrinsic motion is thus obtained by setting to zero the result of substituting these Lie derivatives in place of the corresponding Lagrangian variations in the more general variation formula (52), which gives

$$\sum_d \int {}^{(d)}\!d\overline{\mathcal{S}} \left\{ {}^{(d)}\overline{\boldsymbol{j}}^\mu\, \vec{\xi}\pounds\, \boldsymbol{A}_\mu + \frac{1}{r!}\, {}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots}\, \vec{\xi}\pounds\, \boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots} + \frac{1}{2}\, {}^{(d)}\overline{\boldsymbol{T}}^{\mu\nu}\, \vec{\xi}\pounds\, \boldsymbol{g}_{\mu\nu} \right\} = 0. \tag{71}$$

The requirement that this should hold for any choice of ξ^μ evidently implies that the tangentiality conditions (57) for the surface fluxes ${}^{(d)}\overline{\boldsymbol{j}}^\mu$ and ${}^{(d)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu}$ must be supplemented by an analogous d-surface tangentiality condition for the surface stress momentum energy tensor ${}^{(d)}\overline{\boldsymbol{T}}^{\mu\nu}$, which must satisfy

$${}^{(d)}\!\perp^\mu{}_\nu\, {}^{(d)}\overline{\boldsymbol{T}}^{\nu\rho} = 0. \tag{72}$$

(as again one would expect anyway from the consideration that it depends just on the relevant internal d-surface fields). This allows (70) to be written out in the form

$$\sum_d \int {}^{(d)}\!d\overline{\mathcal{S}} \Bigg\{ \xi^\rho \Bigg(\boldsymbol{F}_{\rho\mu}\,{}^{(d)}\bar{\boldsymbol{j}}^\mu + \frac{1}{r!}\boldsymbol{F}^{\{r+1\}}{}_{\rho\mu\nu\ldots}\,{}^{(d)}\bar{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots} - {}^{(d)}\overline{\nabla}_\mu\,{}^{(d)}\overline{\boldsymbol{T}}{}^\mu_\rho - \boldsymbol{A}_\rho\,{}^{(d)}\overline{\nabla}_\mu\,{}^{(d)}\bar{\boldsymbol{j}}^\mu - \frac{1}{(r-1)!}\boldsymbol{A}^{\{r\}}{}_{\rho\nu\ldots}\,{}^{(d)}\overline{\nabla}_\mu\,{}^{(d)}\bar{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots} \Bigg) + {}^{(d)}\overline{\nabla}_\mu \left(\xi^\rho \left(\boldsymbol{A}_\rho\,{}^{(d)}\bar{\boldsymbol{j}}^\mu + \frac{1}{(r-1)!}\boldsymbol{A}^{\{r\}}{}_{\rho\nu\ldots}\,{}^{(d)}\bar{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots} + {}^{(d)}\overline{\boldsymbol{T}}{}^\mu{}_\rho \right)\right) \Bigg\} = 0, \tag{73}$$

in which the final contribution is a pure surface divergence that can be dealt with using Green's theorem as before. Using the results (61) and (62) of the analysis of the consequences of gauge invariance and proceeding as in their derivation above, one sees that the condition for (73) to hold for an arbitrary field ξ^μ is that, on each $(p-1)$-brane of the complex, the dynamical equations

$${}^{(p)}\overline{\nabla}_\mu\,{}^{(p)}\overline{\boldsymbol{T}}{}^\mu{}_\rho = {}^{(p)}\boldsymbol{f}_\rho, \tag{74}$$

should be satisfied for a total force density ${}^{(p)}\boldsymbol{f}_\rho$ given by

$${}^{(p)}\boldsymbol{f}_\rho = {}^{(p)}\bar{\boldsymbol{f}}_\rho + {}^{(p)}\check{\boldsymbol{f}}_\rho, \tag{75}$$

where ${}^{(p)}\check{\boldsymbol{f}}_\rho$ is the contribution of the contact force exerted on the p-surface by other members of the brane complex, which takes the form

$${}^{(p)}\check{\boldsymbol{f}}_\rho = \sum_{d=p+1} {}^{(d)}\lambda_\mu\,{}^{(d)}\overline{\boldsymbol{T}}{}^\mu{}_\rho, \tag{76}$$

while the other force density contribution ${}^{(p)}\bar{\boldsymbol{f}}_\rho$ represents the effect of the external background fields, which is given by

$${}^{(p)}\bar{\boldsymbol{f}}_\rho = \boldsymbol{F}_{\rho\mu}\,{}^{(p)}\bar{\boldsymbol{j}}^\mu + \frac{1}{r!}\boldsymbol{F}^{\{r+1\}}{}_{\rho\mu\nu\ldots}\,{}^{(p)}\bar{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots}. \tag{77}$$

As before, the summation "over $d = p + 1$" in (76) is to be understood as consisting of a contribution from each of the p-branes attached to the $(p-1)$-brane under consideration, where for each such attached p-brane, ${}^{(d)}\lambda_\mu$ denotes the (uniquely defined) unit tangent vector on its $(p+1)$-dimensional support surface that is directed normally towards the p-dimensional support surface of the boundary $(p-1)$-brane.

The first of the background force contributions in (77) is of course the Lorentz type force density resulting from the effect of the electromagnetic field on the surface current. For the case of an ordinary current 3-vector $\boldsymbol{F}^{\{r+1\}}{}_{\rho\mu\nu}$, the other contribution in (77) will just be the Joukowsky type force density (of the kind responsible for the lift on an aerofoil) resulting from the Magnus effect, which acts in the case of a "global" string (Davis and Shellard, 1989b; Vilenkin and Vachaspati, 1987) through not in the case of a string of the "local" type for which the relevant vorticity flux ${}_{(p)}\overline{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu}$ will be zero. As with the conservation laws (61) and (62), so also the explicit force density balance law expressed by (74) can alternatively be expressed in terms of the corresponding Dirac distributional stress momentum energy and background force density tensors, ${}_{(d)}\hat{\boldsymbol{T}}^{\mu\nu}$ and ${}_{(d)}\hat{\boldsymbol{f}}_{\mu}$, which are given for each $(d-1)$-brane in terms of the corresponding regular surface stress momentum energy and background force density tensors ${}_{(d)}\overline{\boldsymbol{T}}^{\mu\nu}$ and ${}_{(d)}\overline{\boldsymbol{f}}_{\mu}$ by

$$ {}_{(d)}\hat{\boldsymbol{T}}^{\mu\nu} = \|\boldsymbol{g}\|^{-1/2} \int {}_{(d)}d\overline{\mathcal{S}}\, {}_{(d)}\overline{\boldsymbol{T}}^{\mu\nu}\, \delta^n[x - x\{\sigma\}] \tag{78} $$

and

$$ {}_{(d)}\hat{\boldsymbol{f}}_{\mu} = \|\boldsymbol{g}\|^{-1/2} \int {}_{(d)}d\overline{\mathcal{S}}\, {}_{(d)}\overline{\boldsymbol{f}}_{\mu}\, \delta^n[x - x\{\sigma\}]. \tag{79} $$

The equivalent—more formally obvious but less explicitly meaningful—distributional version of the force balance law (74) takes the form

$$ \nabla_{\mu}\left(\sum_{d} {}_{(d)}\hat{\boldsymbol{T}}^{\mu}{}_{\rho}\right) = \hat{\boldsymbol{f}}_{\rho}, \tag{80} $$

where the total Dirac distributional force density is given in terms of the electromagnetic current distributions (64) and the (generalized vorticity) flux distributions (66) by

$$ \hat{\boldsymbol{f}}_{\rho} = \boldsymbol{F}_{\rho\mu} \sum_{d} {}_{(d)}\hat{\boldsymbol{j}}^{\mu} + \frac{1}{r!}\boldsymbol{F}^{\{r+1\}}{}_{\rho\mu\nu\ldots} \sum_{d} {}_{(d)}\hat{\boldsymbol{j}}_{\{r\}}{}^{\mu\nu\ldots}. \tag{81} $$

It is again left as an optional exercise for readers who are adept in the use of Dirac distributions to show that the system (74), (76), and (77) is obtainable from (80) and (81) by substituting (64), (66), (78), and (79).

As an immediate corollary of (74), it is to be noted that for any vector field ℓ^{μ} that generates a continuous symmetry of the background spacetime metric, i.e., for any solution of the Killing equations

$$ \nabla_{(\mu}\ell_{\nu)} = 0, \tag{82} $$

one can construct a corresponding surface momentum or energy density current

$$ {}_{(p)}\overline{\boldsymbol{P}}^{\mu} = {}_{(p)}\overline{\boldsymbol{T}}^{\mu\nu}\ell_{\nu}, \tag{83} $$

that will satisfy

$$ {}_{(p)}\overline{\nabla}_{\mu}\, {}_{(p)}\overline{\boldsymbol{P}}^{\mu} = \sum_{d=p+1} {}_{(d)}\lambda_{\mu}\, {}_{(d)}\overline{\boldsymbol{P}}^{\mu} + {}_{(p)}\overline{\boldsymbol{f}}_{\mu}k^{\mu}. \tag{84} $$

In typical applications for which the n-dimensional background spacetime can be taken to be flat there will be n independent translation Killing vectors which alone (without recourse to the further $n(n-1)/2$ rotation and boost Killing vectors of the Lorentz algebra) will provide a set of relations of the form (84) that together provide the same information as that in the full force balance equation (74) or (80).

13. THE EQUATION OF EXTRINSIC MOTION

Rather than the distributional version (80), it is the explicit version (74) of the force balance law that is directly useful for calculating the dynamic evolution of the brane support surfaces. Since the relation (80) involves n independent components whereas the support surface involved in only p-dimensional, there is a certain redundancy, which results from the fact that if the virtual displacement field ξ^μ is tangential to the surface in question it cannot affect the action. Thus if ${}_{(p)}\perp^\mu_\nu \xi^\nu = 0$, the condition (71) will be satisfied as a mere identity—provided of course that the field equations governing the internal fields of the system are satisfied. It follows that the nonredundent information governing the extrinsic motion of the p-dimensional support surface will be given just by the orthogonally projected part of (74). Integrating by parts, using the fact that, by (7) and (18), the surface gradient of the rank-$(n-p)$ orthogonal projector ${}_{(p)}\perp^\mu_\nu$ will be given in terms of the second fundamental tensor ${}_{(p)}\boldsymbol{K}_{\mu\nu}{}^\rho$ of the p-surface by

$$ {}_{(p)}\overline{\nabla}_\mu\, {}_{(p)}\perp^\nu_\rho = -{}_{(p)}\boldsymbol{K}_{\mu\nu}{}^\rho - {}_{(p)}\boldsymbol{K}_\mu{}^\rho{}_\nu, \tag{85} $$

it can be seen that the extrinsic equations of motion obtained as the orthogonally projected part of (74) will finally be expressible by

$$ {}_{(p)}\overline{\boldsymbol{T}}^{\mu\nu}\, {}_{(p)}\boldsymbol{K}_{\mu\nu}{}^\rho = {}_{(p)}\perp^\rho{}_\mu\, {}_{(p)}\boldsymbol{f}^\mu. \tag{86} $$

It is to be emphasized that the formal validity of the formula that has just been derived is not confined to the variational models on which the above derivation is based, but also extends to dissipative models (involving effects such as external drag by the background medium (Carter *et al.*, 1994; Garriga and Sakellariadou, 1993; Vilenkin, 1991) or mutual resistance between independent internal currents). The condition that even a nonconservative macroscopic model should be compatible with an underlying microscopic model of conservative type requires the existence (representing to averages of corresponding microscopic quantities) of appropriate stress momentum energy density and force density fields satisfying (86).

The ubiquitously applicable formula (86) is interpretable as being just the natural higher generalization of "Newton's law" (equating the product of mass with acceleration to the applied force) in the case of a particle. The surface stress momentum energy tensor, ${}_{(p)}\overline{\boldsymbol{T}}^{\mu\nu}$, generalizes the mass, and the second fundamental tensor, ${}_{(p)}\boldsymbol{K}_{\mu\nu}{}^\rho$, generalizes the acceleration.

The way this works out in the 1-dimensional case of a "pure" point particle (i.e., a monopole) of mass $\boldsymbol{m}$, for which the Lagrangian is given simply by ${}^{(1)}\mathcal{L} = -\boldsymbol{m}$, is as follows. The 1-dimensional energy tensor will be obtained in terms of the unit tangent vector $\boldsymbol{u}^\mu$ ($\boldsymbol{u}^\mu\, \boldsymbol{u}_\mu = -1$) as ${}_{(1)}\overline{\boldsymbol{T}}^{\mu\nu} = \boldsymbol{m}\,\boldsymbol{u}^\mu\, \boldsymbol{u}^\nu$, and in this zero-brane case, the first fundamental tensor will simply by given by ${}_{(1)}\eta^{\mu\nu} = -\boldsymbol{u}^\mu\, \boldsymbol{u}^\nu$, so that the second fundamental tensor will be obtained in terms of the acceleration $\dot{\boldsymbol{u}}^\mu = \boldsymbol{u}^\nu \nabla_\nu\, \boldsymbol{u}^\mu$ as ${}_{(1)}\boldsymbol{K}_{\mu\nu}{}^\rho = \boldsymbol{u}_\mu\, \boldsymbol{u}_\nu\, \dot{\boldsymbol{u}}^\rho$. Thus (86) can be seen to reduce in the case of a particle simply to the usual familiar from $\boldsymbol{m}\,\dot{\boldsymbol{u}}^\rho = {}_{(1)}\perp^\rho{}_\mu\, {}_{(1)}\boldsymbol{f}^\mu$.

The familiar electromagnetic example of the Faraday–Lorentz force exerted on a charged point particle (i.e., a zero-brane) by an ordinary Maxwellian field is the simplest example of the effect of the important special case of what (in view of the proverbial complementarity of "brain versus brawn") may conveniently be termed the relevant "brawn field." For a generic $(p-1)$ brane, with worldsheet dimension p, the corresponding brawn field is defined to be a Ramond type gauge r form $\boldsymbol{A}^{\{r\}}{}_{\mu\nu\ldots}$ whose index number r is equal to the worldsheet dimension, i.e., for which $r = p$. In this case the corresponding generalized vorticity flux on the brane must evidently be given by an expression of the form

$$\overline{\boldsymbol{j}}_{\{p\}}{}^{\mu\nu\ldots} = e_{\{p\}}\, {}^{(p)}\mathcal{E}^{\mu\nu\ldots}, \tag{87}$$

for some proportionality factor $e_{\{p\}}$. Moreover, provided that this brawn source flux is confined to the d-dimensional brane worldsheet, so that the right hand side of the flux conservation law (62) vanishes, this proportionality factor must have vanishing worldsheet gradient,

$${}^{(p)}\overline{\nabla}_\nu e_{\{p\}} = 0, \tag{88}$$

so that $e_{\{p\}}$ will have a fixed value. The coefficient $e_{\{p\}}$ will thus be interpretable as a brawn charge coupling constant characterizing the p-brane. In particular, for the case of a zero-brane (i.e., a point particle) the relevant coupling constant $e_{\{1\}}$ will be interpretable as an ordinary electromagnetic charge. Similarly for a 1-brane (i.e., a string) the relevant (Wess–Zumino type) coupling constant $e_{\{2\}}$ will be interpretable as a measure of the relevant. Kalb–Ramond current circulation round the worldsheet. When the relevant "brawn" field provides the only external force on the brane the orthogonal projection on the right of (86) will be redundant, and the equation of extrinsic motion of the worldsheet will reduce to the explicity form

$${}^{(p)}\overline{\boldsymbol{T}}^{\mu\nu}\, {}^{(p)}\boldsymbol{K}_{\mu\nu\rho} = \frac{e_{\{p\}}}{p!}\, \boldsymbol{F}^{\{d\}}{}_{\rho\sigma\ldots}\, {}^{(p)}\mathcal{E}^{\rho\ldots}, \tag{89}$$

with $d = p + 1$ as before. For the case of a point particle in an electromagnetic field this is just the usual equation of motion provided by the Faraday Lorentz force, while for the case of a string surrounded by a Kalb–Ramond current this is just the equation of motion provided (Carter and Langlois, 1995) by the Joukowski lift force density that is attributable to the familiar Magnus effect. For the source free dynamical equation governing the "brawn field" outside the brane, the simplest possibility is a divergence equation of the familiar form

$$\nabla^{\rho}\, \boldsymbol{F}^{\{d\}}{}_{\rho\sigma\ldots} = 0, \tag{90}$$

which applies both to ordinary Maxwellian electromagnetism and to the standard kind of axion fluid model (Battye and Shellard, 1995, 1996; Carter, 1994b; Dabholkar and Quashnock, 1990).

The possibility that such an effect occurs for the 3-brane of a "brane world" scenario has not yet been given much attention, presumably because a nonzero value for the relevant generalized Wess–Zumino coupling constant $e_{\{4\}}$ would specify a preferred orientation in the worldsheet (due to the pseudo-tensorial, not strictly tensorial, nature of the 4-surface alternating tensor $\mathcal{E}^{\mu\nu\rho\sigma}$) and hence would violate the Z_2 symmetry that is usually postulated in the 5-dimensional scenarios that are most commonly considered (Binetruy *et al.*, 2000; Bowcock *et al.*, 2000; Chamblin and Gibbons, 2000; Chamblin *et al.*, 2000; Langlois *et al.*, 2000; Maartens, 2000; Mennim and Battye, 2000; Shiromizu *et al.*, 2000). However, the consequences of dropping the Z_2 symmetry constraint have recently begun to be a subject of systematic investigation (Davis *et al.*, 2001, Kogan *et al.*, 2000, n.d.). It therefore seems worthwhile to point out that a generalized Wess–Zumino type coupling effect of the type characterized by (89) could provide a plausible underlying mechanism that, subject to (90), would simulate the effect of a discontinuous change of the cosmological "constant" of the "bulk," such as has recently been postulated in bubble type scenarios (Deruelle and Dolezel, 2000; Perkins, 2001) of this less orthodox Z_2 symmetry violating kind. For a brane of codimension 1, i.e., when the backgound dimension is $n = d = p + 1$, the external "brawn" field $\boldsymbol{F}^{\{d+1\}}{}_{\rho\sigma\ldots}$ must evidently be proportional to the background measure tensor $\varepsilon_{\rho\sigma\ldots}$, with a proportionality factor that must be uniform over any region where the source free field equation (90) is satisfied, so that for a $(d-1)$-brane in a $(d+1)$-dimensional bulk we shall have $\boldsymbol{F}^{\{d+1\}}{}_{\mu\nu\ldots} = \boldsymbol{F}^{\{d+1\}}\, \varepsilon_{\mu\nu\ldots}$ with a "brawn" field pseudo-scalar $\boldsymbol{F}^{\{d+1\}}$ that has constant value (giving a stress energy density tensor of the same form as would arise from a cosmological constant proportional to $|\boldsymbol{F}|^2$) which will give rise to a force density with uniform magnitude proportional to the product $e_{\{d\}}\, \boldsymbol{F}^{\{d+1\}}$. Thus using the unit normal $\lambda_\mu = (d!)^{-1}\, \varepsilon_{\mu\nu\ldots}\, {}^{(d)}\mathcal{E}^{\nu\ldots}$ (with $d = 4$ in the usual brane world case) to construct the (symmetric) second fundamental form ${}^{(d)}\boldsymbol{K}_{\mu\nu} = {}^{(d)}\boldsymbol{K}_{\mu\nu}{}^{\rho}\lambda_\rho$, it can be seen that the equation of motion (89) will be expressible in this case as

$$ {}^{(d)}\overline{\boldsymbol{T}}^{\mu\nu}\, {}^{(d)}\boldsymbol{K}_{\mu\nu} = e_{\{d\}}\, \boldsymbol{F}^{\{d+1\}}. \tag{91}$$

(In a more elaborate treatment allowing for the active role of the brane as a source for the "brawn" field, this constant product $e_{\{d\}}\, \boldsymbol{F}^{\{d+1\}}$ would need to be replaced by a constant proportional to the resulting surface discontinuity in $(\boldsymbol{F}^{\{d+1\}})^2$, and if self-gravitation were also taken into account then (as will be discussed in more detail elsewhere) the tensor ${}^{(d)}\boldsymbol{K}_{\mu\nu}$ would also be continuous and its value in (91) would need to be replaced by the mean of its values on the two sides.)

14. PERTURBATIONS AND EXTRINSIC CHARACTERISTIC EQUATION

Two of the most useful formulae for the analysis of small perturbations of a string or higher brane worldsheet are the expressions for the infinitesimal Lagrangian (comoving) variation of the first and second fundamental tensors in terms of the corresponding comoving variation $\underset{\mathrm{L}}{\delta}\, \boldsymbol{g}_{\mu\nu}$ of the metric (with respect to the comoving reference system). For the first fundamental tensor one easily obtains

$$\underset{\mathrm{L}}{\delta}\, \eta^{\mu\nu} = -\eta^{\mu\rho}\eta^{\mu\sigma}\underset{\mathrm{L}}{\delta}\, \boldsymbol{g}_{\rho\sigma}, \qquad \underset{\mathrm{L}}{\delta}\, \eta^{\mu}{}_{\nu} = \eta^{\mu\rho}\perp^{\sigma}{}_{\nu}\underset{\mathrm{L}}{\delta}\, \boldsymbol{g}_{\rho\sigma} \tag{92}$$

and, by substituting this in the defining relation (15), the corresponding Lagrangian variation of the second fundamental tensor is obtained (Carter, 1993) as

$$\underset{\mathrm{L}}{\delta}\, \boldsymbol{K}_{\mu\nu}{}^{\rho} = \perp^{\rho}{}_{\lambda}\eta^{\sigma}{}_{\mu}\eta^{\tau}{}_{\nu}\underset{\mathrm{L}}{\delta}\, \Gamma_{\sigma}{}^{\lambda}{}_{\tau} + \left(2\perp^{\sigma}{}_{(\mu}\boldsymbol{K}_{\nu)}{}^{\tau\rho} - \boldsymbol{K}_{\mu\nu}{}^{\sigma}\eta^{\tau\rho}\right)\underset{\mathrm{L}}{\delta}\, \boldsymbol{g}_{\sigma\tau}, \tag{93}$$

where the Lagrangian variation of the connection (22) is given by the well known formula

$$\underset{\mathrm{L}}{\delta}\Gamma_{\sigma}{}^{\lambda}{}_{\tau} = \boldsymbol{g}^{\lambda\rho}\left(\nabla_{(\sigma}\underset{\mathrm{L}}{\delta}\boldsymbol{g}_{\tau)\rho} - \tfrac{1}{2}\nabla_{\rho}\underset{\mathrm{L}}{\delta}\boldsymbol{g}_{\sigma\tau}\right). \tag{94}$$

Since we are concerned here only with cases for which the background is fixed in advance so that the Eulerian variation d_{E} will vanish in (67), the Lagrangian variation of the metric will be given just by its Lie derivative with respect to the infinitesimal displacement vector field ξ^{μ} that generates the displacement of the worldsheet under consideration, i.e., we shall simply have

$$\underset{\mathrm{L}}{\delta}\, \boldsymbol{g}_{\sigma\tau} = 2\nabla_{(\sigma}\xi_{\tau)}. \tag{95}$$

It then follows from (94) that the Lagrangian variation of the connection will be given by

$$\underset{\mathrm{L}}{\delta}\Gamma_{\sigma}{}^{\lambda}{}_{\tau} = \nabla_{(\sigma}\nabla_{\tau)}\xi^{\lambda} - \mathcal{R}^{\lambda}{}_{(\sigma\tau)\rho}\xi^{\rho}, \tag{96}$$

where $\mathcal{R}^{\lambda}{}_{\sigma\tau\rho}$ is the background Riemann curvature (which will be negligible in typical applications for which the lengthscales characterizing the geometric features of interest will be small compared with those characterizing any background spacetime curvature). The Lagrangian variation of the first fundamental tensor is thus finally obtained in the form

$$\underset{\mathrm{L}}{\delta}\, \eta^{\mu\nu} = -2\eta_{\sigma}{}^{(\mu}\overline{\nabla}^{\nu)}\xi^{\sigma}, \tag{97}$$

while that of the second fundamental tensor is found to be given by

$$\underset{\mathrm{L}}{\delta}\boldsymbol{K}_{\mu\nu}{}^{\rho} = \perp^{\rho}{}_{\lambda}\big(\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)}\xi^{\lambda} - \eta^{\sigma}{}_{(\mu}\eta^{\tau}{}_{\nu)}\mathcal{R}^{\lambda}{}_{\sigma\tau\rho}\xi^{\rho} - \boldsymbol{K}^{\sigma}{}_{(\mu\nu)}\overline{\nabla}_{\sigma}\xi^{\lambda}\big)$$
$$+ \big(2\perp^{\sigma}{}_{(\mu}\,\boldsymbol{K}_{\nu)\tau}{}^{\rho} - \boldsymbol{g}^{\rho}{}_{\tau}\,\boldsymbol{K}_{\mu\nu}{}^{\sigma}\big)(\nabla_{\sigma}\xi^{\tau} + \overline{\nabla}^{\tau}\xi_{\sigma}). \tag{98}$$

It is instructive to apply the foregoing formulae to the case of a free pure brane worldsheet, meaning one for which there is no external force contribution so that the equation of extrinsic motion reduces to the form

$$\overline{\boldsymbol{T}}^{\mu\nu}\,\boldsymbol{K}_{\mu\nu}{}^{\rho} = 0. \tag{99}$$

On varying the relation (99) using (98) in conjunction with the orthogonality property (72) and the unperturbed equation (99) itself, the equation governing the propagation of the infinitesimal displacement vector is obtained in the form

$$\perp^{\rho}{}_{\lambda}\overline{\boldsymbol{T}}^{\mu\nu}\big(\overline{\nabla}_{\mu}\overline{\nabla}_{\nu}\xi^{\lambda} - \mathcal{R}^{\lambda}{}_{\mu\nu\sigma}\xi^{\sigma}\big) = -\boldsymbol{K}_{\mu\nu}{}^{\sigma}\underset{\mathrm{L}}{\delta}\overline{\boldsymbol{T}}^{\mu\nu}. \tag{100}$$

In the simplest case, for which there are no internal fields, (100) constitutes the complete system of dynamical equations, which take an explicit form (Battye and Carter, 1995) that can be shown (Battye and Carter, 2000) to be directly obtainable by application of the variation principle to the second order perturbation of the relevant Dirac–Goto–Nambu action. However, in the generic case, the extrinsic perturbation equation (100) will by itself be only part of the complete system of perturbation equations governing the evolution of the brane, the remaining equations of the system being those governing the evolution of whatever surface current (Carter, 1989b) and other relevant internal fields on the supporting worldsheet may be relevant. The perturbations of such fields are involved in the source term on the right of (100), whose explicit evaluation depends on the specific form of the relevant currents or other internal fields. However, it is not necessary to know the specific form of such internal fields for the purpose just of deriving the characteristic velocities of propagation of the extrinsic propagations represented by the displacement vector ξ^{μ}, so long as they contribute to the source term on the right of the linearized perturbation equation (100) only at first differential order, so that the characteristic velocities will be completely determined by the first term on the left of (100) which will be the only second differential order contribution. It is apparent from (100) that under these conditions the equation for the characteristic tangent covector χ_{μ} say will be given independently of any details of the surface currents or other internal fields simply (Carter, 1990) by

$$\overline{\boldsymbol{T}}^{\mu\nu}\chi_{\mu}\chi_{\nu} = 0. \tag{101}$$

(It can be seen that the unperturbed surface stress momentum energy density tensor $\overline{\boldsymbol{T}}^{\mu\nu}$ plays the same role here as that of the unperturbed metric tensor $\boldsymbol{g}^{\mu\nu}$ in the analogous characteristic equation for the familiar case of a massless background spacetime field, as exemplified by electromagnetic or gravitational radiation.)

ACKNOWLEDGMENTS

I wish to thank B. Allen, C. Barrabès, R. Battye, A.-C. Davis, R. Davis, V. Frolov, G. Gibbons, R. Gregory, T. Kibble, D. Langlois, K. Maeda, X. Martin, P. Peter, T. Piran, D. Polarski, M. Sakellariadou, P. Shellard, P. Townsend, N. Turok, T. Vachaspati, and A. Vilenkin for many stimulating or clarifying discussions.

REFERENCES

Achúcarro, A., Evans, J., Townsend, P. K., and Wiltshire, D. L. (1987). Super p-branes. *Physics Letters* **198B**, 441–446.

Arodz, H., Sitarz, A., and Wegrzyn, P. (1991). *Acta Physica Polonica B* **22**, 495; 23 (1992) 53.

Barrabés, C., Boisseau, B., and Sakellariadou, M. (1994). Gravitational effects on domain walls with curvature corrections. *Physical Review D* **49**, 2734–2739.

Bars, I. and Pope, C. N. (1988). Anomalies in super p-branes. *Classical and Quantum Gravity* **5**, 1157–1168.

Battye, R. A. and Carter, B. (1995). Gravitational perturbations of relativistic membranes and strings. *Physics Letters B* **357**, 29–35, hep-ph/9508300.

Battye, R. A. and Carter, B. (2000). Second order Lagrangian and symplectic current for gravitationally perturbed Dirac–Goto–Nambu strings and branes. *Classical and Quantum Gravity* **17**, 3325–3334, hep-th/9811075.

Battye, R. A. and Shellard, E. P. S. (1995). String radiative backradiation. *Physical Review Letters* **75**, 4354–4357, astro-ph/9408078.

Battye, R. A. and Shellard, E. P. S. (1996). Radiative backreaction on global strings. *Physical Review D* **53**, 1811, hep-ph/9508301.

Ben-Ya'acov, U. (1992). Unified dynamics of quantum vortices. *Nuclear Physics B* **382**, 597–615.

Binetruy, P., Defffayet, C., and Langlois, D. (2000). Non-cosmological cosmology from a brane universe. *Nuclear Physics B* **565**, 269–287, hep-th/9905012.

Boisseau, B. and Letelier, P. S. (1992). Cosmic strings with curvature corrections. *Physical Review D* **46**, 1721–1729.

Bowcock, P., Charmousis, C., and Gregory, R. (2000). General brane cosmologies and their global spacetime structure, hep-th/0007177. *Class. Quant. Grav.* **17**, 4745–4764.

Brandenberger, R., Carter, B., Davis, A.-C., and Trodden, M. (1996). Cosmic vortons and particle constraints. *Physical Review D* **54**, 6059–6071, hep-ph/9605382.

Capovilla, R. and Guven, J. (1995a). Geomety of deformations of relativistic membranes. *Physical Review D* **51**, 6736, gr-qc/9411060.

Capovilla, R. and Guven, J. (1995b). Large deformations of relativistic membranes: A generalisation of the Raychaudhuri equations. *Physical Review D* **52**, 1072, gr-qc/9411061.

Carter, B. (1989a). Duality relation between charged elastic strings and superconducting cosmic strings. *Physics Letters B* **224**, 61–66.

Carter, B. (1989b). Stability and characteristic propagation speeds in superconducting cosmic and other string models. *Physics Letters B* **228**, 466–470.

Carter, B. (1990). Covariant mechanics of simple and conducting cosmic strings and membranes. In *Formation and Evolution of Cosmic Strings*, G. Gibbons, S. Hawking, and T. Vachaspati, eds., Cambridge University Press, Cambridge, pp. 143–178.

Carter, B. (1992a). Outer curvature and conformal geometry of an imbedding. *Journal of Geometry and Physics* **8**, 53–88.

Carter, B. (1992b). Basic brane theory. *Journal of Classical and Quantum Gravity* **9**, 19–33.

Carter, B. (1993). Perturbation dynamics for membranes and strings governed by Dirac–Goto–Nambu action in curved space. *Physical Review D* **48**, 4835–4838.

Carter, B. (1994a). Equations of motion of a stiff geodynamic string or higher brane. *Classical and Quantum Gravity* **11**, 2677–2692.

Carter, B. (1994b). Axionic vorticity variational formulation for relativistic perfect fluids. *Classical and Quantum Gravity* **11**, 2013–2130.

Carter, B. (1995). Dynamics of cosmic strings and other brane models. In *Formation and Interactions of Topological Defects* (NATO ASI B349), R. Brandenberger and A.-C. Davis, eds., Plenum, New York, pp. 304–348.

Carter, B. and Battye, R. (1998). Nondivergence of gravitational self interactions for Nambu–Goto strings. *Physics Letters B* **430**, 49–53, hep-th/9803012.

Carter, B. and Gregory, R. (1995). Curvature corrections to dynamics of domain walls. *Physical Review D* **51**, 5839–5846, hep-th/9410095.

Carter, B. and Langlois, D. (1995). Kalb–Ramond coupled vortex fibration model for relativistic fluid dynamics. *Nuclear Physics B* **454**, 402–424, hep-th/9611082.

Carter, B., Sakellariadou, M., and Martin, X. (1994). Cosmological expansion and thermodynamic mechanisms in cosmic string dynamics. *Physical Review D* **50**, 682–699.

Chamblin, A. and Gibbons, G. (2000). Supergravity on the brane. *Physical Review Letters* **84**, 1090–1093, hep-th/9909130.

Chamblin, A., Hawking, S. W., and Real, H. S. (2000). Brane world black holes. *Physical Review D* **61**, 065007, hep-th/9909205.

Copeland, E., Haws, D., Kibble. T. W. B., Mitchel, D., and Turok, N. (1988). Monopoles connected by strings. *Nuclear Physics B* **298**, 458–492.

Dabholkar, A. and Quashnock, J. M. (1990). Pinning down the axion. *Nuclear Physics B* **333**, 815–832.

Davis, A.-C., Davis, S. C., Perkins, W. B., and Vernon, I. R. (2001). Brane world phenology and the Z_2 symmetry, hep-ph/0008132. *Phys. Lett. B* **504**, 254–261.

Davis, R. L. and Shellard, E. P. S. (1989a). Cosmic vortons. *Nuclear Physics B* **323**, 209–2024.

Davis, R. L. and Shellard, E. P. S. (1989b). Global strings and superfluid vortices. *Physical Review Letters* **63**, 2021–2024.

Deruelle, N. and Dolezel, T. (2000). Brane versus shell cosmologies in Einstein and Einstein–Gauss–Bonnet theories, gr-qc/0004021. *Phys. Rev. D* **62**, 103502.

Dirac, P. A. M. (1962). An extensible model of the electron. *Proceedings of the Royal Society of London A* **268**, 57–67.

Eisenhart, L. P. (1926). *Riemannian Geometry*, Princeton University Press, Princeton, reprinted 1960.

Garriga, J. and Sakellariadou, M. (1993). Effects of friction on cosmic strings. *Physical Review* **48**, 2502–2515, gr-qc/9307008.

Gherhgetta, T. and Shaposhnikov, M. (2000). Localising gravity on a string-like defect in six dimensions, hep-th/0004014. *Phys. Rev. Lett.* **85**, 240–243.

Gregory, R. (1988). The effective action for a cosmic string. *Physics Letters B* **206**, 199–204.

Gregory, R. (1993). Effective actions for bosonic topological defects. *Physical Review D* **43**, 520–525.

Gregory, R., Haws, D., and Garfinkle, D. (1991). Dynamics of domain walls and strings. *Physical Reivew D* **42**, 343–345.

Hartely, D. H. and Tucker, R. W. (1990). In *Geometry of Low Dimensional Manifolds, Vol. 1,* S. Donaldson and C. Thomas, eds., Cambridge University Press, Cambridge, L.M.S. Lecture Note Series Vol. 150.

Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Spacetime*, Cambridge University Press, Cambridge.

Howe, P. S. and Tucker, R. W. (1977). A locally supersymmetric and reparametrisation invariant action for a spinning membrane. *Journal of Physics A* **10**, L155–L162.

Kibble, T. W. B. (1976). Topology of cosmic domains and strings. *Journal of Physics A* **9**, 1387–1398.

Kogan, I. I., Mouslopoulos, S., Papazoglou, A., Ross, G. C., and Santiago, J. (2000). Three three-brane universe: New phenomenology for the new millenium? *Nuclear Physics B* **584**, 313–328, hep-ph/9912552.

Kogan, I. I., Mouslopoulos, S., Papazoglou, A., and Ross, G. G. (n.d.). Multi-brane worlds and modification of gravity at large scales, hep-th/0006030.

Langlois, D., Maartens, R., and Wands, D. (2000). Gravitational waves from inflation on the brane, hep-th/0006007. *Phys. Lett. B* **489**, 259–267.

Larsen, A. L. (1993). A note on dispersive versus non-dispersive strings. *Classical and Quantum Gravity* **10**, L35–L38.

Letelier, P. S. (1990). Nambu bubbles with curvature corrections. *Physical Review D* **41**, 1333–1335.

Maartens, R. (2000). Cosmological dynamics on the brane, hep-th/0004166. *Phys. Rev. D* **62**, 084023.

Maeda, K. I. and Turok, N. (1988). Finite width corrections to the Nambu action for the Nielsen–Olesen string. *Physics Letters B* **202**, 376–384.

Manton, N. S. (1938). Topology in the Weinberg–Salam theory. *Physical Review D* **28**, 2019–2026.

Martin, X. and Vilenkin, A. (1996). Gravitational background from hybrid topological defects. *Physical Review Letters* **77**, 2879, astro-ph/9606022.

Mennim, A., and Battye, R. A. (2000). Cosmological expansion on a dilatonic brane world, hep-th/0008192. *Class. Quant. Grav.* **18**, 2171–2194.

Nambu, Y. (1977). String-like configurations in the Weinberg–Salam theory. *Nuclear Physics B* **130**, 505–515.

Penrose, R. and Rindler, W. (1984). *Spinors and Space-Time*, Cambridge University Press, Cambridge.

Perkins, W. B. (2001). Colliding bubble worlds. gr-qc/0010053. *Phys. Lett. B* **504**, 28–32.

Polyakov, A. (1986). Fine structure on strings. *Nuclear Physics B* **268**, 406–412.

Sakellariadou, M. (1991). Radiation of Nambu–Goldstone bosons from infinitely long strings. *Physical Review D* **44**, 3767–3773.

Schouten, J. A. (1954). *Ricci Calculus*, Springer, Heidelberg.

Shellard, E. P. S. (1990). Axion strings and domain walls. In *Formation and Evolution of Cosmic Strings*, G. Gibbons, S. Hawking, and T. Vachaspati, eds., Cambridge University Press, Cambridge, pp. 107–115.

Shiromizu, T., Maeda, K., and Sasaki, M. (2000). The Einstein equations on the 3-brane world. gr-qc/9910076. *Phis. Rev. D* **62**, 024012.

Sikivie, P. (1982). Axions, domain walls, and the early univese. *Physical Review Letters* **48**, 1156–1159.

Silveira, V. and Maia, M. D. (1993). Topological defects and corections to the Nambu action. *Physics Letters A* **174**, 280–288.

Stachel, J. (1980). Thickenning the string: The perfect string dust. *Physical Review D* **21**, 2171–2181.

Vachaspati, T. and A. Achúcarro (1991). Semilocal cosmic strings. *Physical Review D* **44**, 3067–3071.

Vachaspati, T. and Barriola, M. (1992). A new class of defects. *Physical Review Letters* **69**, 1867–1872.

Vilenkin, A. (1991). Cosmic string dynamics with friction. *Physical Review D* **43**, 1060–1062.

Vilenkin, A. and Everett, A. E. (1982). Cosmic strings and domain walls in models with Goldstone and pseudo-Goldstone bosons. *Physical Review Letters* **48**, 1867–1870.

Vilenkin, A. and Vachaspati, T. (1987). Radiation of Goldstone bosons from cosmic strings. *Physical Review D* **35**, 1138–1140.

Witten, E. (1985). Superconducting strings. *Nuclear Physics B* **249**, 557–592.